

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \leq t < 10 \\ t & t \geq 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t - 2)u_{10}(t)$

$$\begin{aligned} \text{Solve } 3y'' + y' + y &= 2u_4(t) + (t - 2)u_{10}(t), \\ y(0) &= 0, y'(0) = 0. \end{aligned}$$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t - 2)u_{10}(t))$$

$$\text{Thm: } \mathcal{L}(u_c(t)f(t - c)) = e^{-cs}\mathcal{L}(f(t)).$$

Thus $\mathcal{L}(u_c(t)f(t)) = \underline{\hspace{4cm}}$.

$$\begin{aligned} 3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) \\ = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t + 8)) \end{aligned}$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$\begin{aligned} y &= 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) \\ &\quad + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right) \end{aligned}$$

$$y = u_4(t)f(t - 4) + u_{10}h(t - 10) + 8u_{10}f(t - 10)$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$ and $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = A(3s^2 + s + 1) + (Bs + C)s$$

$$0s^2 + 0s + 1 = (3A + B)s^2 + (A + C)s + A$$

$$0 = 3A + B, 0 = A + C, 1 = A$$

Hence $A = 1, B = -3A = -3, C = -A = -1$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\frac{1}{3})-\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3(s+\frac{1}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$\begin{aligned}
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{36})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6} \cdot \frac{6}{\sqrt{11}} \cdot \frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}} \cdot \frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)
\end{aligned}$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct} \mathcal{L}^{-1}(F(s))$

$$\begin{aligned}
&= 1 + e^{-\frac{1}{6}t} \mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}t} \mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right) \\
&= 1 - e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6} t - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6} t
\end{aligned}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$$

$$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$$

$$0 = 3A+B, 0 = A+3D+C, 0 = A+D, 1 = D.$$

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$,
 $B = -3A = 3$.

$$\begin{aligned}
\frac{1}{s^2[3s^2+s+1]} &= \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]} \\
&= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} \\
&= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}} \cdot \frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} \\
&= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}} \cdot \frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} \\
&= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}
\end{aligned}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$\begin{aligned}
&= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right) \\
&= -1 + t + e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6} t - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6} t
\end{aligned}$$

Hence the final answer is

$$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$\begin{aligned}
&= u_4(t)[1 - e^{-\frac{1}{6}(t-4)} \cos \frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-4)} \sin \frac{\sqrt{11}}{6}(t-4)] + \\
&u_{10}[-1 + t - 10 + e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)] \\
&+ 8u_{10}[1 - e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)]
\end{aligned}$$