The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$

2.) Use the fact that the LaPlace Transform is linear:

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

3.) Use thm to change this equation into an algebraic equation:

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$

3.5) Substitute in the initial values:

$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$s^2 + 3s + 4$: $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an $s^2 - a^2$ or an $s^2 + a^2$ or an $(s-a)^2$, so it must be an $(s-a)^2 + b^2$.

Hence we will complete the square:

$s^2 + 3s + \underline{\quad} - \underline{\quad} + 4 = (s + \underline{\quad})^2 - \underline{\quad} + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s - a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to use $\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2}) = e^{at}cosbt$
and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2}) = e^{at}sinbt$

$5s + 21 = 5(s + \frac{3}{2}) - \frac{15}{2} + 21 = 5(s + \frac{3}{2}) + \frac{27}{2}$

$= 5(s + \frac{3}{2}) + [\frac{27}{2}\sqrt{\frac{4}{7}}]\sqrt{\frac{7}{4}} = 5(s + \frac{3}{2}) + [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5(s+\frac{3}{2})+[\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$= 5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] + \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4}) = \mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] + \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$= 5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}) + \frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$

$= 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t + \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$

Hence $y(t) = 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t + \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$.