

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$$

2.) Use the fact that the LaPlace Transform is linear:

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$$

3.) Use thm to change this equation into an algebraic equation:

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$$

3.5) Substitute in the initial values:

$$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$$

$$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$$

$$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}\left(\frac{5s+21}{s^2+3s+4}\right)$$

$$y = \mathcal{L}^{-1}\left(\frac{5s+21}{s^2+3s+4}\right)$$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$$s^2 + 3s + 4: b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$$

Hence $s^2 + 3s + 4$ does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

$s^2 + 3s + 4$ is not an $s^2 - a^2$ or an $s^2 + a^2$ or an $(s-a)^2$, so it must be an $(s-a)^2 + b^2$.

Hence we will complete the square:

$$s^2 + 3s + \underline{\quad} - \underline{\quad} + 4 = (s + \underline{\quad})^2 - \underline{\quad} + 4$$

$$\text{Hence } \frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$$

Must now consider the numerator. We need to look like $s-a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at} \cos bt$$

$$\text{and/or } \mathcal{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at} \sin bt$$

$$5s + 21 = 5\left(s + \frac{3}{2}\right) - \frac{15}{2} + 21 = 5\left(s + \frac{3}{2}\right) - \frac{27}{2}$$

$$= 5\left(s + \frac{3}{2}\right) - \left[\frac{27}{2} \sqrt{\frac{4}{7}}\right] \sqrt{\frac{7}{4}} = 5\left(s + \frac{3}{2}\right) - \left[\frac{27}{\sqrt{7}}\right] \sqrt{\frac{7}{4}}$$

$$\text{Hence } \frac{5s+21}{s^2+3s+4} = \frac{5\left(s+\frac{3}{2}\right) - \left[\frac{27}{\sqrt{7}}\right] \sqrt{\frac{7}{4}}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}$$

$$= 5\left[\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right] - \frac{27}{\sqrt{7}} \left[\frac{\sqrt{\frac{7}{4}}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right]$$

$$\text{Thus } \mathcal{L}^{-1}\left(\frac{5s+21}{s^2+3s+4}\right) = \mathcal{L}^{-1}\left(5\left[\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right] - \frac{27}{\sqrt{7}} \left[\frac{\sqrt{\frac{7}{4}}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right]\right)$$

$$= 5\mathcal{L}^{-1}\left(\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right) - \frac{27}{\sqrt{7}} \mathcal{L}^{-1}\left(\frac{\sqrt{\frac{7}{4}}}{\left(s+\frac{3}{2}\right)^2 + \frac{7}{4}}\right)$$

$$= 5e^{-\frac{3}{2}t} \cos \sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}} e^{-\frac{3}{2}t} \sin \sqrt{\frac{7}{4}}t$$

$$\text{Hence } y(t) = 5e^{-\frac{3}{2}t} \cos \sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}} e^{-\frac{3}{2}t} \sin \sqrt{\frac{7}{4}}t.$$