The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$$

2.) Use the fact that the LaPlace Transform is linear:

$$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$$

3.) Use thm to change this equation into an algebraic equation:

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$$

3.5) Substitute in the initial values:

$$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2+b^2$ OR if you should factor and use partial fractions

s^2+3s+4: $b^2-4ac = 3^2-4(1)(4) = 9-16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an s^2-a^2 or an s^2+a^2 or an $(s-a)^2$, so it must be an $(s-a)^2+b^2$.

Hence we will complete the square:

$s^2+3s+\_\_\_ - \_\_\_ + 4 = (s+\_\_\_)^2 - \_\_\_ + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s-a=s+\frac{3}{2}$ or $b=\sqrt{\frac{7}{4}}$ in order to use
$\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2})=e^{at}cosbt$
and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2})=e^{at}sinbt$

$$5s+21=5(s+\tfrac{3}{2})-\tfrac{15}{2}+21=5(s+\tfrac{3}{2})-\tfrac{27}{2}$$

$$=5(s+\tfrac{3}{2})-[\tfrac{27}{2}\sqrt{\tfrac{4}{7}}]\sqrt{\tfrac{7}{4}}=5(s+\tfrac{3}{2})-[\tfrac{27}{\sqrt{7}}]\sqrt{\tfrac{7}{4}}$$

Hence $\frac{5s+21}{s^2+3s+4}=\frac{5(s+\frac{3}{2})-[\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$$=5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}]-\frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})=\mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}]-\frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$$=5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}})-\frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$$

$$=5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t-\frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$$

Hence $y(t)=5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t-\frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t.$