Thursday, April 9, 2010

Theorem 3: K tame, V Seifert Matrix, P presentation matrix for $H_1(\tilde{X})$

$$\Rightarrow P = V^T - tV$$
$$= V - tV^T$$

Corollary: $\Delta(t) = \det(V^T - tV)$

Lemma 14: Given $\{a_1, \dots, a_{2g}\}$ basis for $H_1(M^\circ)$, $\exists$ basis $\{m_1, \dots, m_{2g}\}$ dual to $\{a_1, \dots, a_{2g}\}$ for $H_1(S^3 \setminus M)$ $\ni$

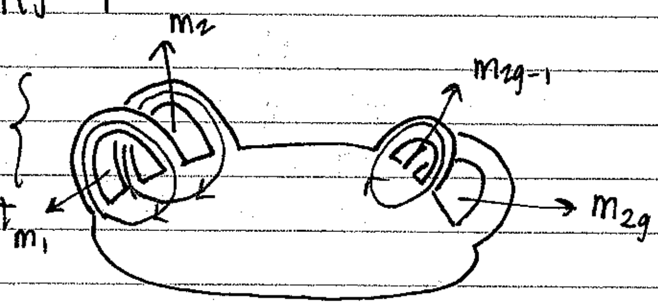
$$lk(a_i, m_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Proof: Classifications of surfaces:

If we know g & # of ∂ components, then we know the # of handles &

If occur in pairs, $\partial = 1$

comp knot



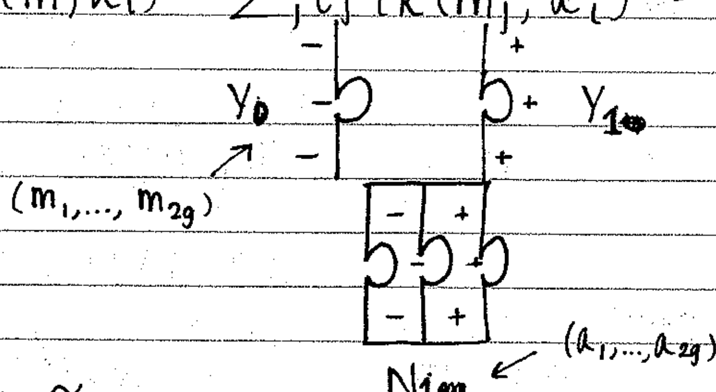
Proof of Theorem 3: Let N be the bicollar of S

Let $\{a_1, \dots, a_{2g}\}$ be a basis for $H_1(M^\circ)$, & let $V = (lk(a_i, a_j^+))_{2g \times 2g}$

Let $\{m_1, \dots, m_{2g}\}$ be a basis for $H_1(S^3 \setminus M)$ as in Lemma 14.

$m \in H_1(S^3 \setminus M) \Rightarrow m = \sum_j c_j m_j$

$lk(m, a_i) = \sum_j c_j \, lk(m_j, a_i) = c_i$



$H_1(\tilde{X}) = \{m_1, \dots, m_{2g} \mid a^- = t a^+\}$

as Λ-module.

$a_k^- = \sum c_j m_j$ in Y_0 where $c_j = lk(a_k^-, a_j)$

$a_k^+ = \sum d_j m_j$ where $d_j = lk(a_k^+, a_j)$

Thus, $\sum lk(a_k^-, a_j) m_j = t \sum lk(a_k^+, a_j) m_j$

(Note: $a_i^- = t^{1} a_i^+$ in Y_1)

($\curvearrowleft$ in Y_0, $i = 1, \dots, 2g$)

$lk(a_k^-, a_j) = lk(a_k, a_j^+)$ (moving up one level via ambient isotopy)

$\sum_j [lk(a_k, a_j^+) - t\, lk(a_j, a_k^+)] m_j$

$= \sum (V_{kj} - t V_{jk}) m_j$

$P = V - tV^T$.

If we exchange + & − sides, $P = V^t - tV$. ▨

<u>Corollary 7</u>: $\Delta(t^{-1}) = \det(V^T - t^{-1} V)$

$\doteq \det(tV^T - V)$

equality up to mult. by $\pm t^k$ $\doteq \det(V - tV^T)$

$= \Delta(t)$

$\Delta(1) = \det(V - V^T) = \det(D) = \pm 1$.

If $q(t)$ is an Alexander Polynomial, then $q(t^{-1}) = q(t) \cdot (\pm t^k)$ for some knot

The converse is also true. $q(1) = 1$

<u>Def</u>: $\deg\left(\sum_{i=k}^{l} n_i t^i\right) = l - (k)$ if $n_l, n_k \neq 0$

<u>Ex</u>: $\deg(\Delta(t)) \leq 2g(K)$

<u>Ex 12</u>: crossing # of $K \geq \deg(\Delta(t)) + 1$

<u>Note</u>: Read/do Ex. 17.

Knots 6_1, $9_{46} \longrightarrow \Delta(t) = 2 - 5t + 2t^2$

$H_1(S^3 \setminus 6_1) = \Lambda / (2 - 5t + 2t^2)$

$H_1(S^3 \setminus 9_{46}) = \Lambda/(2t-1) \times \Lambda/(2t^{-1} - 1)$

For $\widetilde{S^3 \setminus 9_{46}}$

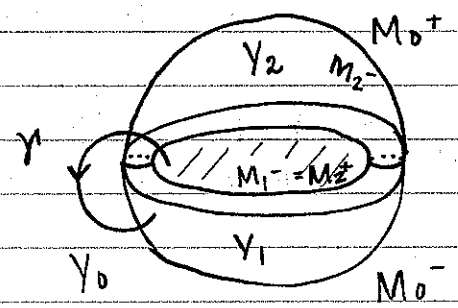
$$P = \begin{bmatrix} 2t-1 & 0 \\ 0 & 2t^{-1}-1 \end{bmatrix}$$

$$\det(P) = 4 - 2t - 2t^{-1} + 1$$
$$\doteq 5 - 2t - 2t^{-1}$$
$$\doteq 2 - 5t + 2t^2$$

<u>Section</u> D : <u>Torsion</u> <u>Invariant</u> $\underline{H_1}(\tilde{X}_K)$

<u>EX</u>: $\tilde{X}_3$

$S^3 \setminus O_1$



3-fold cover

$$\tilde{X}_3 \cup N(K) = \Sigma_3 = \text{cyclic branched cover (3-fold)}$$

3 copies of $S^3 \setminus N(K)$

1 copy of $N(K)$

$$H_1(\tilde{X}_3) = H_1(\Sigma_3) \oplus \mathbb{Z}$$

from γ

<u>Theorem 1</u>: $H_1(\Sigma_2)$ as a $\mathbb{Z}$-module (i.e. as a group) has a presentation matrix of $V + V^T$

<u>Proof</u>: $\Sigma_2 = Y_0 \cup Y_1 \cup N(K) = \bigcup^2 (S^3 \setminus M) \cup N(K)$

$H_1(\Sigma_2) = (m_1^0, \ldots, m_{2g}^0, m_1^1, \ldots, m_{2g}^1)$

$\sum_i v_{ij} \alpha_j^0 = \sum_i v_{ji} \alpha_j^1$

$\sum_i v_{ij} \alpha_j^1 = \sum_i v_{ji} \alpha_j^0$

$\vec{m^i} = (m_1^i, \ldots, m_{2g}^i)$

$\vec{m^0}\, V = \vec{m^1}\, V^T$

$\vec{m^1}\, V = \vec{m^0}\, V^T$

$\vec{m_1}\,(V - V^T) = \vec{m_0}\,(V^T - V)$

$\det(V - V^T) = \pm 1 \Rightarrow$ invertible.

$\overrightarrow{m^1} = -\overrightarrow{m^0}$

$m_i^1 = -m_i^0$

$\overrightarrow{m^0} V = -\overrightarrow{m^0} V^T$

$\overrightarrow{m^0}(V+V^T) = 0$

$H_1(\Sigma_2) = \{m_1^0, \ldots, m_{2g}^0 \mid \overrightarrow{m^0}(V+V^T) = 0\}$

$P = V + V^T$. ∎

<u>Other Properties</u>

① Mod 2, $\Delta(1) = \Delta(-1) = \det(V - V^T) = \pm 1$

$\Rightarrow \Delta(1)$ is odd

② $H_1(\Sigma_2)$ as a $\mathbb{Z}$-module implies P has $\mathbb{Z}$-entries

$P \xrightarrow[\substack{\text{row ops. except} \\ \text{row}_i \leftrightarrow a\,\text{row}_i}]{} E \leftarrow$ echelon form

③ $\det P = \pm \det E$

<u>Ex</u>: $\begin{bmatrix} & & * \\ & & * \\ 0 & \cdots & 3\ * \\ 0 & \cdots\cdots & 4 \end{bmatrix}$ $\quad * \in \{0, 1, \ldots, 4-1\}$

$0, \mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}$

$|H_1(\Sigma_2)| = |\det(P)| = |V + V^T|$

<u>Cor</u>: $|H_1(\Sigma_2)|$ is finite & odd

$H_1(\Sigma_K) = \left\{ \begin{matrix} m_j^i, & i = 0, \ldots, k-1 \\ & j = 1, \ldots, 2g \end{matrix} \;\middle|\; \begin{matrix} \overrightarrow{m^i} V^T = \overrightarrow{m^{i+1}} V \\ \text{for } i = 0, \ldots, k-1 \mod k \end{matrix} \right\}$

<u>Suppose V is invertible</u>

$$\left.\begin{aligned} \overrightarrow{\alpha_1} &= \overrightarrow{\alpha_0} V^T V^{-1} \\ \overrightarrow{\alpha_2} &= \overrightarrow{\alpha_1}(V^T V^{-1}) = \overrightarrow{\alpha_0}(V^T V^{-1})^2 \\ &\vdots \\ \overrightarrow{\alpha_k} &= \overrightarrow{\alpha_0}(V^T V^{-1})^k \end{aligned}\right\} \alpha = m$$

$$= \left\{ m_1^0, \ldots, m_{2g}^0 \mid \overrightarrow{\alpha_0}\,[\underbrace{(V^T V^{-1})^k - I}_{P}] \right\}$$