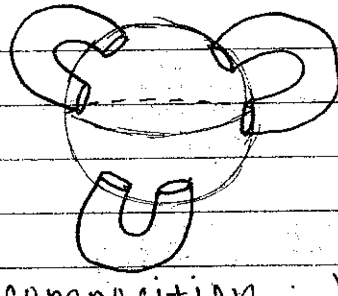
Tuesday, April 20, 2010

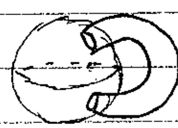
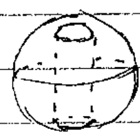
Handlebody of genus g

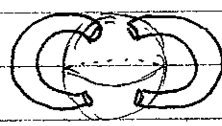
Let U =  Let $\Sigma = \partial U$

Heegaard Decomposition : $Y = U_1 \underset{\Sigma}{\cup} U_2$

$U_1 = U_2$ = a handle body of genus g

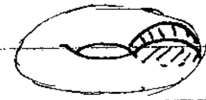
Ex: S^3 =  $\underset{S^2}{\cup}$ 

 $\underset{T^2}{\cup}$ 

 $\cup$ 

⋮

genus g decomposition of S^3

Ex: $L(p,q)$ =  $\underset{T^2}{\cup}$ 

$m \longmapsto pL + qM$

① glue in neighborhood of meridinal disk using $m \longmapsto pL + qM$.

② glue in remaining 3-ball.

Note: Heegaard decomposition of $L(p,q)$ depends only on curves m, $pL + qM$.

$L(p,q)$ =  

drill out tubular nbhd of unknotted arc

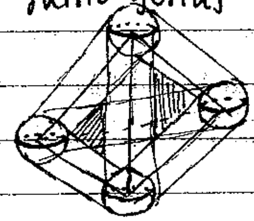
can easily increase genus of H.D.

Theorem 2.1 (Singer 1933): Let Y be an oriented closed 3 mfld. Then Y admits a Heegaard decomposition.

Proof: Start w/ a triangulation of Y.

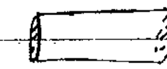
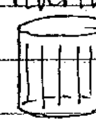
same ∂
same genus

0-handles $\cup$ 1-handles = handlebody



3-handles $\cup$ 2-handles = handlebody

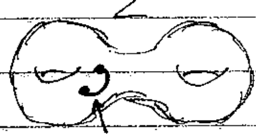
dual triangulation

3-h	2-h	1-h	0-h
0-handles:	1-handles	2-handles	3-handles
 $D^3 \times \{0\}$	 $D^2 \times D^1$	 $D^1 \times D^2$	$\{0\} \times D^3$ 
Add to each vertex	Glue on $D^2 \times \partial D^1$ for each edge	Glue $D^1 \times \partial D^2$ for each face	Glue along ∂D^3 for each tetrahedron

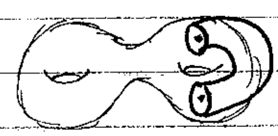
Stabilization of $Y = U_1 \cup U_2$

U_i are genus g handlebodies.

Σ'



drill out unknotted arc = A



$$(U_1 \setminus \mathrm{nbhd}(A)) \underset{\partial \Sigma'}{\cup} (U_2 \cup \mathrm{nbhd}(A))$$

$$g(\Sigma') = g(\Sigma) + 1$$

Note: Removing a handle results in destabilization.

Let $Y = U_1 \cup U_2 = \tilde{U}_1 \cup \tilde{U}_2$

$\Sigma \leftarrow$ genus g $\quad \tilde{\Sigma} \leftarrow$ genus $\tilde{g}$

Then for k large enough, the $k-g$-fold stabilization of $U_1 \cup_\Sigma U_2$ is diffeo w/ $(k-\tilde{g})$-fold stable of $\tilde{U}_1 \cup_{\tilde{\Sigma}} \tilde{U}_2$

Note: Stabilization / Destabilization is like Reidemeister moves for manifolds.

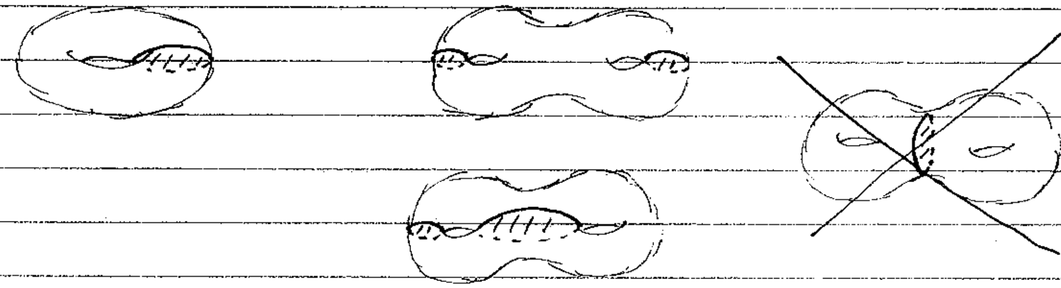
If f : 3-manifolds $\longrightarrow X$ does not change under stab/destab, then f is an invariant of 3-manifolds.

Def: A set of attaching circles $\{\gamma_1, \dots, \gamma_g\}$ for U, a genus g handlebody is a collection of closed embedded curves in $\partial U = \Sigma_g \ni$

① $\gamma_i \cap \gamma_j = \emptyset \ \forall i \neq j$

② $\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected
i.e. $\{[\gamma_1], \dots, [\gamma_g]\}$ are L.I. in $H_1(\Sigma, \mathbb{Z})$

③ γ_i bound disjoint embedded disks in U.



Let (Σ_g, U_1, U_2) be a genus g. H.D. for Y. A compatible Heegaard diagram is given by $(\Sigma_g, \underbrace{\alpha_1, \dots, \alpha_g}_{\text{attaching circles for } U_1}, \underbrace{\beta_1, \dots, \beta_g}_{\text{for } U_2})$