Thursday, March 11, 2010
Section 7D

$$\pi_1(X) \longrightarrow \text{Alex. Inv. } H_1(\tilde{X})$$
$$p_*: \pi_1(\tilde{X}) \longrightarrow \pi_1(X)$$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \times I \arrow[ur, "H"] \arrow[r, dashed] & X \end{tikzcd} \qquad \begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \times I \arrow[ur, dashed, "\tilde{H}"] \arrow[r] & X \end{tikzcd}$$

p_* well-defined $\qquad$ p_* is 1-1 since H lifts

Def: Suppose $\tilde{X}$ is the universal abelian cover of X. Then

$$p_*(\pi_1(\tilde{X})) = [\pi_1(X), \pi_1(X)]$$ "commutator"
$$= \langle aba^{-1}b^{-1} \mid a,b \in \pi_1(X) \rangle \text{ (not abelian)}$$
$$= C = \text{smallest normal subgroup} \ni \pi_1(X)/C \text{ is abelian}$$

Note: $\tilde{X}$ is a regular covering of X since C is a normal subgroup.

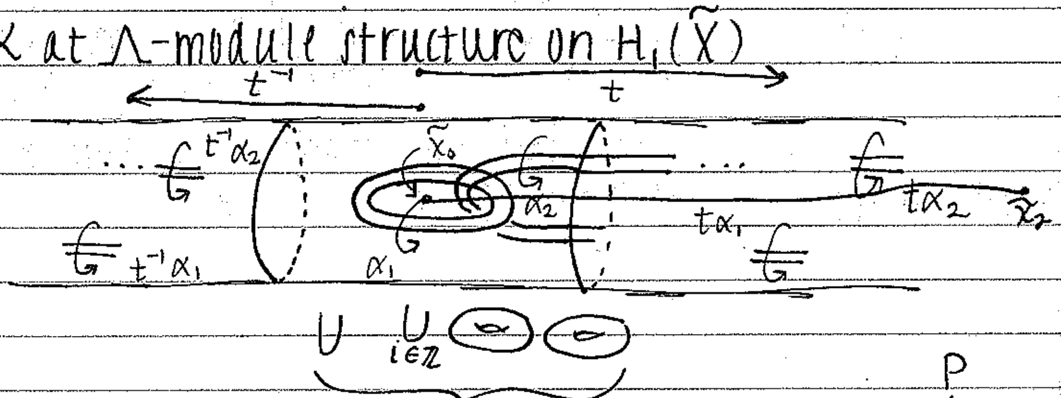
$p_*: \pi_1(\tilde{X}) \longrightarrow [\pi_1(X), \pi_1(X)]$ is a group isomorphism

$$\overline{p}_*: \pi_1(\tilde{X}) \big/ [\pi_1(X), \pi_1(X)] = \cancel{H_1(X)} \longrightarrow C/[C,C]$$

is a group isomorphism

Question: What about the Λ-module structure??

A look at Λ-module structure on $H_1(\tilde{X})$



$p \downarrow$

$x_0 \quad (X, x_0) \quad \tilde{x}_i \in p^{-1}(x_0)$

universal abelian cover

$$\mathrm{Aut}(\tilde{X}, p) \cong \pi_1(X) \Big/ \underbrace{p_*(\pi_1(\tilde{X}))}_{\text{normal}}$$
$$\cong H_1(X)$$
$$\cong \mathbb{Z}^d \quad \text{where } d = \#\text{ of components of link}$$

<u>Note</u>: Universal abelian cover = infinite cyclic cover
$\iff$ K is a 1-component knot.

<u>Note</u>: $\tilde{X}$ regular, $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$

$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = g\, p_*(\pi_1(\tilde{X}, \tilde{x}_0))\, g^{-1} \quad \text{(conjugate)}$$
for some $g \in \pi_1(X)$
$$= p_*(\pi_1(\tilde{X}_1, \tilde{x}_0)) \text{ since } \tilde{X} \text{ is regular.}$$

$$\begin{tikzcd}
{[\lambda]} \arrow[ddd] \arrow[rrr, bend left] & & & p_*[\lambda] = [c] \arrow[ddd, dashed] \\
& H(\tilde{X}) \arrow[r, "\bar{P}_*", "\cong"'] \arrow[d, "\tau_*"', "\cong"] & C/[C,C] \arrow[d, dashed] & \\
& H_1(\tilde{X}) \arrow[r, "\bar{P}_*", "\cong"'] & C/[C,C] & \\
{[t\lambda]} \arrow[rrr, bend right] & & & {[xcx^{-1}]}
\end{tikzcd}$$

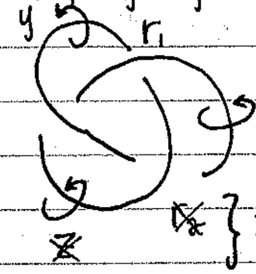
<u>Def 7D1</u>: Define $t : \mathcal{C}/[\mathcal{C},\mathcal{C}] \longrightarrow \mathcal{C}/[\mathcal{C},\mathcal{C}]$ by
$$t([c]) = [xcx^{-1}]$$
for any $x \xrightarrow{\text{abel}} [1] \in H_1(X)$

<u>Claim</u>: t is well-defined

① $xcx^{-1} \in \mathcal{C}$ since $xcx^{-1} = (xcx^{-1}c^{-1}) \cdot c$

② Suppose $d \in [c]$. Suppose $x, y \longrightarrow [1] \in H_1(X)$.
Then $cd^{-1} \in [\mathcal{C}, \mathcal{C}]$.
We must show that $ydy^{-1} \in [xcx^{-1}]$, i.e.
$$(xcx^{-1})(ydy^{-1})^{-1} \in [\mathcal{C}, \mathcal{C}] \quad \text{(Proof: Exercise r).}$$

Ex: $\pi_1(S^3 \setminus 3_1) = \langle x, y \mid yxyx^{-1}y^{-1}x^{-1} \rangle$



$\rightsquigarrow$ gives Wirtenger presentation

z & r_2 are redundant

$$y(xyx^{-1}y^{-1})x^{-1} = 1$$
$$xyx^{-1}y^{-1} = y^{-1}x$$

Rewriting, we have

$$\langle x, y \mid yxyx^{-1}y^{-1}x^{-1} \rangle = \langle x, yx^{-1} \mid \cdots \rangle$$
$$= \langle x, a \mid axxa\cancel{xx^{-1}}x^{-1}a^{-1}x^{-1} \rangle$$

$a = yx^{-1}$

$ax = y$

$1 \in H_1(X)$, $0 \in C$

$$= \langle x, a \mid a(x^2ax^{-2})(xa^{-1}x^{-1}) \rangle$$

After several steps, we see...

$$C \cong \langle x^k a^{\pm 1} x^{-k} \mid a(x^2ax^{-2})(xa^{-1}x^{-1}), \ldots \rangle$$

so since $x^k a^{\pm 1} x^{-k} \xrightarrow{p_*} t^k \alpha^{\pm 1}$,

$$H_1(\tilde{X}) = (\alpha \mid \alpha + t^2\alpha - t\alpha)$$

as a Λ-module.

Homework 7D4:

$\pi_1(X) \cong \langle x, a_1, \ldots, a_p \mid r_1, \ldots, r_q \rangle$

where $\pi_1(X) \xrightarrow{\text{abel.}} H_1(X)$

$x \longrightarrow 1$

$a_i \longrightarrow 0$ [i.e. $a_i \in C$]

$X = S^3 \setminus K$ where K is a 1-comp. knot

Note: X tame $\Rightarrow \pi_1$ is finitely presented & $p = q$.

Proof: Use Wirtenger Presentation.

Exercise 5: $C = [\pi_1(X), \pi_1(X)]$ is generated by $x^k a_i^{\pm 1} x^{-k}$ where $i = 1, \ldots, p$ & $k \in \mathbb{Z}$ & r_i can be written as products of $x^k a_i^{\pm 1} x^{-k}$ $i = 1, \ldots, p$, $k \in \mathbb{Z}$

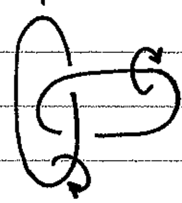
To get to λ-module, use

$$x^k a_i^{\pm 1} x^{-k} \longrightarrow \pm t^k \alpha_i$$

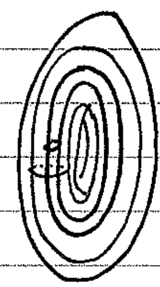
Moving on to links...

<u>Question</u>: How do we find the universal abelian cover for $S^3 \setminus L$?

<u>Easiest Example</u>: Hopf Link



$H_1(X) = \mathbb{Z} \times \mathbb{Z}$

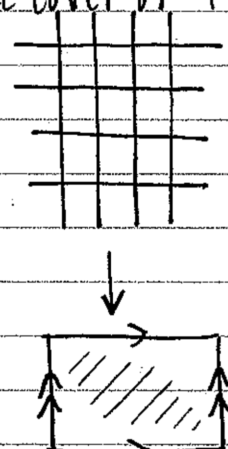


$S^3 \setminus$ Hopf Link $\cong S^1 \times S^1 \times (0,1)$

$\mathbb{R}^2$ = universal cover of T^2

$H_1(\tilde{X}) = 0$

$= \Lambda / \langle 1 \rangle$



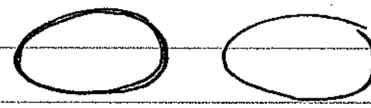
$\times (0,1)$

$\text{Aut}(\tilde{X}) \cong \mathbb{Z}^2$

Alex. poly. $= 1$

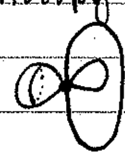
$\times (0,1)$

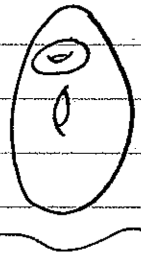
<u>Less Easy Example</u>: Unlink



has same homotopy type as:

sphere





def. retract