Exam 1 Feb. 26, 2009 SHOW ALL WORK
Math 28 Calculus III Either circle your answers or place on answer line.

[14] 1.) Use the chain rule to calculate $D(f \circ g)(s,t)$ where $f : \mathbf{R}^2 \to \mathbf{R}^3$, $f(x,y) = (x, y, e^{xy})$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$, $g(s,t) = (t^2, sin(st))$.

$$D(f \circ g)\big|_{(s,t)} = Df\big|_{g(s,t)} \; Dg\big|_{(s,t)}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ ye^{xy} & xe^{xy} \end{pmatrix}\Bigg|_{(t^2, \sin(s t))} \begin{pmatrix} 0 & 2t \\ t\cos(st) & s(\cos(st)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sin(st)e^{t^2\sin(st)} & t^2 e^{t^2 \sin(st)} \end{pmatrix} \begin{pmatrix} 0 & 2t \\ t\cos(st) & s(\cos(st)) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 & 2t \\ t\cos(st) & s(\cos(st)) \\ t^3\cos(st)e^{t^2\sin(st)} & 2t\cos(st)e^{t^2\sin(st)} + st^2\cos(st)e^{t^2\sin(st)} \end{pmatrix}}$$

Answer: $D(f \circ g)(s,t) =$ ______________________

Will also accept $\vec{a} = (2,0)$

[14] 2a.) Suppose $f(x,y) = e^{xy}$. Approximate $f(1.9, 0.1)$ by finding a best linear approximation to $f$ at an appropriate $\mathbf{x} = \mathbf{a}$. Let $\vec{a} = (1.9, 0)$

$$f(1.9, 0) = e^0 = 1$$

$$Df(1.9, 0) = \left(ye^{xy} \quad xe^{xy}\right)\Big|_{(1.9,0)}$$

$$= (0, \ 1.9)$$

$$f(x,y) \sim h(x,y) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

$$= 1 + (0, 1.9)\begin{pmatrix} x - 1.9 \\ y - 0 \end{pmatrix}$$

$$= 1 + 1.9y$$

$$f(1.9, 0.1) \sim h(1.9, 0.1) = 1 + (1.9)(0.1)$$

$$= 1 + 0.19 = 1.19$$

Answer 2a: $f(1.9, 0.1) \sim$ $\underline{1.19}$

[6] 2b.) $D_{(3,4)} f(10,2) =$ $\underline{\left(46/5\right) e^{20}}$ where $f(x,y) = e^{xy}$.

$$\nabla f(\cancel{1}0, 2) = \left(ye^{xy} \quad xe^{xy}\right)\Big|_{(10,2)}$$

$$= \left(2e^{20}, \ 10e^{20}\right)$$

$$D_{\frac{(3,4)}{\|(3,4)\|}} f(10,2) = \left(2e^{20}, 10e^{20}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{6}{5}e^{20} + \frac{40}{5}e^{20}$$

$$= \frac{46}{5}e^{20}$$

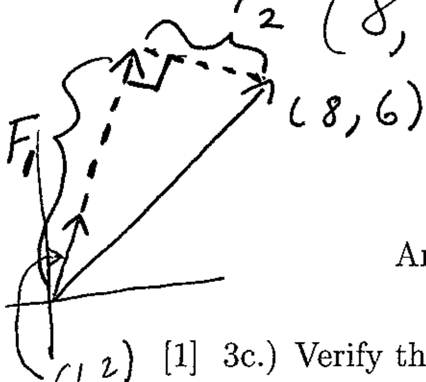
[5] 3a.) $proj_{(1,2)}(8,6) =$ $(4, 8)$

$$\frac{(1,2)\cdot(8,6)}{(1,2)\cdot(1,2)}(1,2) = \frac{8+12}{1+4}(1,2)$$

$$= \frac{20}{5}(1,2) = 4(1,2)$$

[4] 3b.) Suppose that a force $\mathbf{F} = (8,6)$ is acting on an object moving parallel to the vector (1, 2). Decompose the vector (8, 6) into a sum of vectors $\mathbf{F_1}$ and $\mathbf{F_2}$ where $\mathbf{F_1}$ points along the direction of motion and $\mathbf{F_2}$ is perpendicular to the direction of motion.

$F_2$ $(8,6) - (4,8) = (4,-2)$




Answer 3b: $\mathbf{F_1} =$ $(4, 8)$, $\mathbf{F_2} =$ $(4, -2)$

[1] 3c.) Verify that $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$

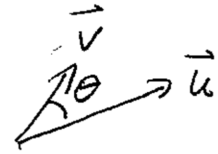
$$(4,8) + (4,-2) = (8,6)$$

[4] 3d.) Use the dot product to verify that $\mathbf{F_1}$ and $\mathbf{F_2}$ are perpendicular to each other. Explain how the dot product can be used to verify that two vectors are perpendicular.

$$(4,8)\cdot(4,-2) = 16 - 16 = 0$$

$$\vec{v} \perp \vec{u} \iff \vec{v}\cdot\vec{u} = 0$$

since $\vec{v}\cdot\vec{u} = ||\vec{v}||\ ||\vec{u}|| \cos\theta$




where $\theta$ is the angle btwn $\vec{v}$ & $\vec{u}$

Thus $\vec{v}\cdot\vec{u} = 0 \iff \vec{v} = \vec{0}$ or $\vec{u} = \vec{0}$ or $\cos\theta = 0$

$\cos\theta = 0 \iff \theta = \pi/2$ since $\theta \in [0, \pi]$

[12] 4.) Find the following limit if it exists. If it doesn't exist, state why you know it doesn't exist.

$$lim_{(x,y)\to(0,0)} \left(\frac{2x^2 - y^2}{x^2 + y^2}\right)$$

Approach (0,0) along line $y=0$:

$$\lim_{(x,0)\to(0,0)} \left(\frac{2x^2 - 0^2}{x^2+0^2}\right) = \lim_{x\to 0} \left(\frac{2x^2}{x^2}\right) = 2$$

Approach (0,0) along line $x=0$:

$$\lim_{(0,y)\to(0,0)} \left(\frac{2(0)^2 - y^2}{0^2+y^2}\right) = -1$$

$2 \neq -1 \Rightarrow$ limit DNE

$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{L} \Leftrightarrow$

$f(\vec{x}) \to \vec{L}$ no matter how $\vec{x} \to \vec{a}$.

Thus if we can find two different paths where $\vec{x} \to \vec{a}$, but $f(\vec{x})$ approaches different values for these two paths, then $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ does not exist.

[5] 5.) State the limit definition of differentiable:

$f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{x} = \mathbf{a}$ if the Jacobian matrix $Df(\vec{a})$ exists

and if $\displaystyle \lim_{\vec{x}\to\vec{a}} \frac{\| f(\vec{x}) - [f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})] \|}{\|\vec{x}-\vec{a}\|} = 0$

That is $h : \mathbb{R}^n \to \mathbb{R}$

$h(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$ is a

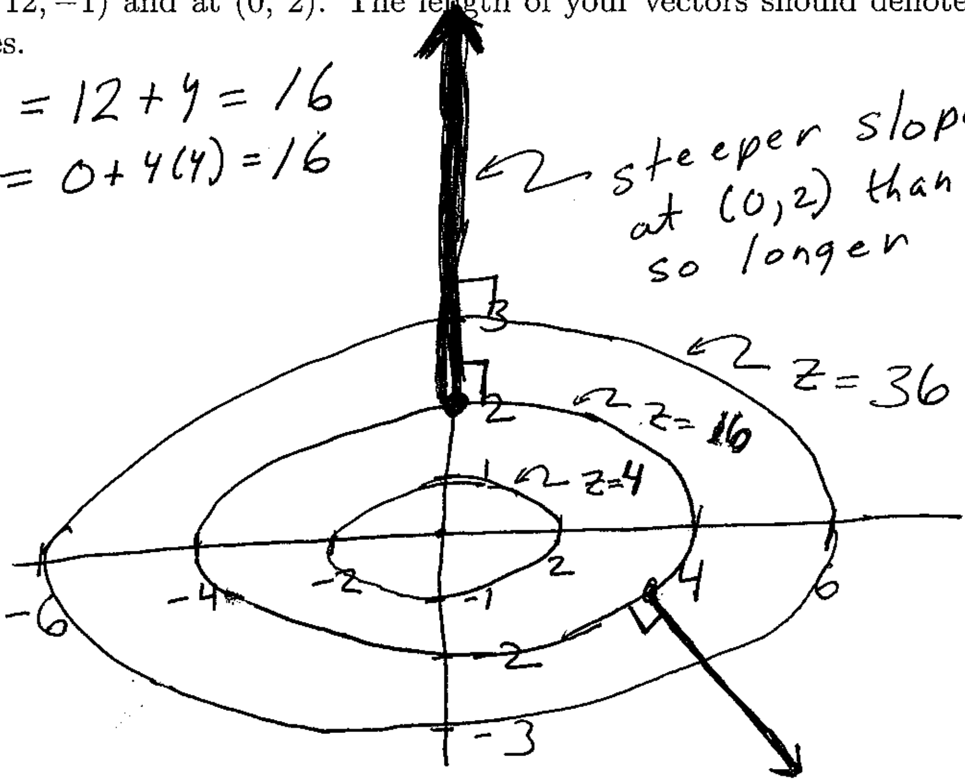
good linear approximation of $f$ near $\vec{a}$

[12] 6a.) Let $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x, y) = x^2 + 4y^2$. Draw several level curves of $f$ (make sure to indicate the height $c$ of each curve). Draw vectors in the direction of the gradient of $f$ at $(\sqrt{12}, -1)$ and at (0, 2). The length of your vectors should denote their relative magnitudes.

$f(\sqrt{12}, -1) = 12 + 4 = 16$

$f(0, 2) = 0 + 4(4) = 16$

steeper slope at $(0,2)$ than $(\sqrt{12}, -1)$ so longer vector



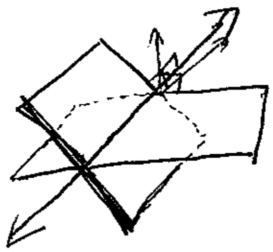

[2] 6b.) Identify the quadric surface in 5a: elliptic paraboloid

To find pt on line find a pt that lies on both planes: $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 10 \\ 4^{-4} & 5^{+2} & -10^{-6} & 20^{-20} \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 10 \\ 0 & 7 & -16 & 0 \end{array}\right]$

$\Rightarrow (5,0,0)$ is a solution

$\Rightarrow (5,0,0)$ is a pt on line

[12] 7.) State the equation for the line of intersection of the planes $2x - y + 3z = 10$ and $4x + 5y - 10z = 20$



$(2,-1,3) \perp$ plane $2x-y+3z=10$

$(4,5,-10) \perp$ plane $4x+5y-10z=20$

Since line of intersection lies in both planes, $(2,-1,3)$ & $(4,5,-10)$ are $\perp$ to line of intersection.

$\Rightarrow$ the direction of the line is $\perp$ to $(2,-1,3)$ & $(4,5,-10)$

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 4 & 5 & -10 \end{vmatrix} = (-5, +32, 14)$$

$\Rightarrow$ line: $\vec{x} = \vec{p} + t\vec{v}$

Answer $(x,y,z) = (5,0,0) + t(-5,+32,14)$

8.) Circle T for True and F for False:

[3] a.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If $f$ is differentiable, then $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exists and is continuous for $i = 1, ..., n$. $\Leftarrow$ T F

[3] b.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exists and is continuous for $i = 1, ..., n$, then $f$ is differentiable at $\mathbf{a}$. T F

[3] c.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$. If $D_{\mathbf{v}}(f)(\mathbf{a})$ exists for all $\mathbf{v}$, then $f$ is differentiable at $\mathbf{a}$. $\Leftarrow$ T F

[3] d.) If $f$ is continuous, then $f$ is differentiable. $\Leftarrow$ T F