Let $f : \mathbf{R}^n \to \mathbf{R}$.

Define $g_j : \mathbf{R} \to \mathbf{R}$ by $g_i(t) = f_i(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n)$.

If g is differentiable at a_j, then the partial derivative of f with respect to x_j is defined by

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}) = g'(x_j) = lim_{h\to 0} \frac{g(x_j+h)-g(x_j)}{h}$$

$$= lim_{h\to 0} \frac{f_i(x_1,...,x_{j-1},x_j+h,x_{j+1},...,x_n)-f_i(x_1,...,x_{j-1},x_j,x_{j+1},...,x_n)}{h}$$

$$= lim_{h\to 0} \frac{f_i(\mathbf{x}+h\mathbf{e_j})-f_i(\mathbf{x})}{h}$$

Ex: $\frac{\partial(x^2y)}{\partial x} =$ $\frac{\partial(x^2y)}{\partial x} =$

The gradient of f is denoted by

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f_1}{\partial x_1}(\mathbf{a}), ..., \frac{\partial f_1}{\partial x_n}(\mathbf{a})\right)$$

Ex: $\nabla x^2 y =$

When is f differentiable (not just partially differentiable)?

Ex: $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x,y) = \begin{cases} 0 & \text{(x, y) = (0, 0)} \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$

$$\frac{\partial f}{\partial x}(0,0) = lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = lim_{h\to 0} \frac{0-0}{h} = 0$$

BUT f is not continuous at (0, 0)!!!!!!!!!

RECALL: f differentiable implies f continuous

Ex: $g : \mathbf{R}^2 \to \mathbf{R}$, $g(x,y) = x + 2y$

$$\frac{\partial g}{\partial x}(0,0) = lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\frac{\partial g}{\partial y}(0,0) = lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = lim_{h\to 0} \frac{2h-0}{h} = 2$$

We will see later that g is differentiable.

Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$ is differentiable. Then

$$lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$$

$$lim_{h\to 0}\frac{f(a+h)-f(a)}{h} - f'(a) = 0$$

$$lim_{h\to 0}\frac{f(a+h)-f(a)-f'(a)h}{h} = 0$$

or equivalently,

$$lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$$

$$lim_{x\to a}\frac{f(x)-f(a)}{x-a} - f'(a) = 0$$

$$lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)} = 0$$

$$lim_{x\to a}\frac{f(x)-[f(a)+f'(a)(x-a)]}{(x-a)} = 0$$

Thus when x is close to a, then $f(x)$ is close to $f(a)+f'(a)[x-a]$

Thus $y = f(a) + f'(a)[x-a]$ is the best linear approximation of f near $x = a$.

Defn: Let V and W be vector spaces. A **linear transformation** from V to W is a function $T : V \to W$ that satisfies the following two conditions. For each $\mathbf{u}$ and $\mathbf{v}$ in V and scalar a,

i.) $T(a\mathbf{u}) = aT(\mathbf{u})$

ii.) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Thm: Let $T : V \to W$ be a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$

Pf: $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$

Thm: Let A be an $m \times n$ matrix. Then the function

$$T : \mathbf{R}^n \to \mathbf{R}^m$$
$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

Thm: If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = [T(\mathbf{e_1})...T(\mathbf{e_n})]$$

Ex: If $T : \mathbf{R}^2 \to \mathbf{R}$ and $T(1,0) = 3$, $T(0,1) = 4$, then

$$T(x,y) = xT(1,0) + yT(0,1) = (3 \;\; 4)\begin{pmatrix} x \\ y \end{pmatrix} = 3x + 4y$$

Defn: Suppose $f : \mathbf{R}^1 \to \mathbf{R}^1$ is differentiable at a. Then

$$lim_{h\to 0}\frac{f(a+h)-f(a)-f'(a)h}{h} = 0$$

$$lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)} = 0$$

$$lim_{x\to a}\frac{f(x)-[f(a)+f'(a)(x-a)]}{(x-a)} = 0$$

Thus when x is close to a, then $f(x)$ is close to $f(a)+f'(a)[x-a]$

Thus $y = f(a)+f'(a)[x-a]$ is the best linear approximation of f near $x = a$.

Hence the tangent line to f at a is $y = f(a)+f'(a)[x-a]$

The tangent line can be written as a constant plus a linear function.

Defn: Suppose $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^m$.

f is said to be **differentiable at a point a** if there exists an open ball V such that $a \in V \subset A$ and a linear function T such that

$$lim_{\mathbf{h}\to \mathbf{0}}\frac{||f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-T(\mathbf{h})||}{||\mathbf{h}||} = 0$$

$$lim_{\mathbf{x}\to \mathbf{a}}\frac{||f(\mathbf{x})-f(\mathbf{a})-T(\mathbf{x}-\mathbf{a})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

Suppose $f : \mathbf{R}^n \to \mathbf{R}$.

Then $T = (b_1 \;\; b_2 \; ... \; b_n)$ and $T\mathbf{x} = (b_1 \;\; b_2 \; ... \; b_n)\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \Sigma b_i x_i$

Also, $T(\mathbf{x}-\mathbf{a}) = (b_1 \;\; b_2 ... \; b_n)\begin{pmatrix} x_1-a_1 \\ x_2-a_2 \\ . \\ . \\ . \\ x_n-a_n \end{pmatrix} = \Sigma b_i(x_i-a_i)$

$$lim_{\mathbf{x}\to \mathbf{a}}\frac{||f(\mathbf{x})-f(\mathbf{a})-T(\mathbf{x}-\mathbf{a})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

$$lim_{\mathbf{x}\to \mathbf{a}}\frac{||f(\mathbf{x})-f(\mathbf{a})-\Sigma b_i(x_i-a_i)||}{||\mathbf{x}-\mathbf{a}||} = 0$$

$$lim_{\mathbf{x}\to \mathbf{a}}\frac{f(\mathbf{x})-f(\mathbf{a})-\Sigma b_i(x_i-a_i)}{||\mathbf{x}-\mathbf{a}||} = 0$$

$y = f(\mathbf{a})+\Sigma b_i(x_i-a_i)$ approximates $y = f(\mathbf{x})$