6.3 Vector Line integrals:

Calc 1 review: Suppose $f : \mathbf{R} \to \mathbf{R}$,

Fundamental Theorem of Calculus: $\int_a^b f'(t)dt = f(b) - f(a)$.

$\int_a^b$(velocity) dt = distance traveled.

$\int_a^b$(rate of change)dt = total change.

Given $f'(t)$, can find $f(b) - f(a)$.

Given velocity, can find distance traveled.

Given rate of change, can find total change.

Calc III: Suppose $f : \mathbf{R}^2 \to \mathbf{R}$,

Given $\partial f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, find $f(\mathbf{p}) - f(\mathbf{q})$.

6.1 vector field integral review:

Let $F(x,y)=(x,y)$, let $x(t)=$

Notation 1: Work definition

$\int_C F\cdot ds=\int_{C_1}F\cdot ds+\int_{C_2}F\cdot ds$

$=\int_0^1 F(\mathbf{x}_1(t))\cdot\mathbf{x}_1'(t)dt+\int_0^1 F(\mathbf{x}_2(t))\cdot\mathbf{x}_2'(t)dt$

$=\int_0^1 F(t,0)\cdot(1,0)dt+\int_0^1 F(1,t)\cdot(0,1)dt$

$=\int_0^1(t,0)\cdot(1,0)dt+\int_0^1(1,t)\cdot(0,1)dt$

$=\int_0^1 tdt+\int_0^1 tdt=2(\frac{1}{2}t^2)|_0^1=1$

Notation 2: differential form

$\int_C F\cdot ds=\int_{C_1}F\cdot ds+\int_{C_2}F\cdot ds$

$=\int_{C_1}F\cdot(dx,dy)+\int_{C_2}F\cdot(dx,dy)$

$=\int_{C_1}(x,y)\cdot(dx,dy)+\int_{C_2}(x,y)\cdot(dx,dy)$

$=\int_{C_1}(xdx+ydy)+\int_{C_2}(xdx+ydy)$

$=\int_0^1 tdt+\int_0^1(1(0)+tdt)=2(\frac{1}{2}t^2)|_0^1=1$

Note: Both of the above methods are algebraically equivalent, so it doesn't matter which notation you use.

Notation 3: Tangent vector (circulation)

$\int_C F \cdot ds = \int_{\mathbf{x}} F(x,y) \cdot T(x,y) ds$ where T is the unit tangent vector to the path $\mathbf{x}$.

$\mathbf{x}_1(t) = (t,0)$, $\mathbf{x}_1'(t) = (1,0)$, $||\mathbf{x}_1'(t)|| = 1$.
Thus $T_1(x,y) = (1,0)$

$\mathbf{x}_2(t) = (1,t)$, $\mathbf{x}_2'(t) = (0,1)$, $||\mathbf{x}_2'(t)|| = 1$.
Thus $T_2(x,y) = (0,1)$

$\int_C F \cdot ds = \int_{\mathbf{x}_1} F \cdot T_1 ds + \int_{\mathbf{x}_2} F \cdot T_2 ds$

$= \int_{\mathbf{x}_1} F(x,0) \cdot (1,0) ds + \int_{\mathbf{x}_2} F(1,y) \cdot (0,1) ds$

$= \int_{\mathbf{x}_1} (x,0) \cdot (1,0) ds + \int_{\mathbf{x}_2} (1,y) \cdot (0,1) ds$

$= \int_{\mathbf{x}_1} x ds + \int_{\mathbf{x}_2} y ds$ [Note: these are scalar line integrals]

$= \int_0^1 t||x_1'(t)||dt + \int_0^1 t||x_2'(t)||dt$

$= \int_0^1 t(1)dt + \int_0^1 t(1)dt = 2(\frac{1}{2}t^2)|_0^1 = 1$

Note: The algebra for notation 3 is slightly messier than for notations 1 and 2, thus this notation is normally used only when it is possible to determine circulation geometrically.

Can use any parametrization for the path $\mathbf{x}$. For example:

$\mathbf{x}_1 : [0,1] \to \mathbf{R}^2$, $\mathbf{x}_1(t) = (t^2, 0)$

$\mathbf{x}_2 : [0,1] \to \mathbf{R}^2$, $\mathbf{x}_2(t) = (t^3, 0)$

Notation 1: Work definition

$\int_C F \cdot ds = \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds$

$= \int_0^1 F(\mathbf{x}_1(t)) \cdot \mathbf{x}_1'(t)dt + \int_0^1 F(\mathbf{x}_2(t)) \cdot \mathbf{x}_2'(t)dt$

$= \int_0^1 F(t^2, 0) \cdot (2t, 0)dt + \int_0^1 F(1, t^3) \cdot (0, 3t^2)dt$

$= \int_0^1 (t^2, 0) \cdot (2t, 0)dt + \int_0^1 (1, t^3) \cdot (0, 3t^2)dt$

$= \int_0^1 2t^3 dt + \int_0^1 3t^5 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Notation 2: differential form

$\int_C F \cdot ds = \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds$

$= \int_{C_1} F \cdot (dx, dy) + \int_{C_2} F \cdot (dx, dy)$

$= \int_{C_1} (x, y) \cdot (dx, dy) + \int_{C_2} (x, y) \cdot (dx, dy)$

$= \int_{C_1} [xdx + ydy] + \int_{C_2} [xdx + ydy]$

$= \int_0^1 t^2(2t)dt + \int_0^1 1(0) + t^3 3t^2 dt = \frac{1}{2}t^4|_0^1 + \frac{1}{2}t^6|_0^1 = 1$

Note: Both of the above methods are algebraically equivalent, so it doesn't matter which notation you use.

Notation 3: Tangent vector (circulation)

$\int_C F \cdot ds = \int_{\mathbf{x}} F(x,y) \cdot T(x,y) ds$ where T is the unit tangent vector to the path $\mathbf{x}$.

$\mathbf{x}_1(t) = (t^2, 0)$, $\mathbf{x}_1'(t) = (2t, 0)$, $||\mathbf{x}_1'(t)|| = 2t$.
Thus $T_1(x,y) = (1,0)$

$\mathbf{x}_2(t) = (1, t^3)$, $\mathbf{x}_2'(t) = (0, 3t^2)$, $||\mathbf{x}_2'(t)|| = 3t^2$.
Thus $T_2(x,y) = (0,1)$

$\int_C F \cdot ds = \int_{\mathbf{x}_1} F \cdot T_1 ds + \int_{\mathbf{x}_2} F \cdot T_2 ds$

$= \int_{\mathbf{x}} F(x,0) \cdot (1,0) ds + \int_{\mathbf{x}} F(1,y) \cdot (0,1) ds$

$= \int_{\mathbf{x}} (x,0) \cdot (1,0) ds + \int_{\mathbf{x}} (1,y) \cdot (0,1) ds$

$= \int_{\mathbf{x}} x ds + \int_{\mathbf{x}} y ds$ [Note: these are scalar line integrals]

$= \int_0^1 t^2 ||x_1'(t)|| dt + \int_0^1 t^3 ||x_2'(t)|| dt$

$= \int_0^1 t^2 (2t) dt + \int_0^1 t^3 3t^2 dt = \frac{1}{2} t^4 |_0^1 + \frac{1}{2} t^6 |_0^1 = 1$

Note: The algebra for notation 3 is slightly messier than for notations 1 and 2, thus this notation is normally used only when it is possible to determine circulation geometrically.

Method 2: 6.2 Green's Theorem

Curve is not closed, so can't use Green's Theorem.

Method 3: 6.3 Claim F has path independent line integrals

Claim F is a gradient field

submethod 1:
$F(x,y) = (x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \nabla f = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$.

I.e, $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

$\int_C F \cdot ds = f(1,1) - f(0,0) = \frac{1}{2} + \frac{1}{2} - 0 = 1$.

submethod 2: Claim $(\nabla \times F)(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^n$

Since $n = 2$, $(\nabla \times F)(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^n$ iff $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, where $F = (M, N)$.

$\frac{\partial y}{\partial x} = 0 = \frac{\partial x}{\partial y}$

Thus can choose any path starting at (0, 0) and ending at (1, 1)

Ex: Let $\mathbf{x} : [0,1] \to \mathbf{R}^2$, $\mathbf{x}(t) = (t,t)$.

$\int_C F \cdot ds = \int_0^1 F(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt$

$= \int_0^1 F(t,t) \cdot (1,1)dt$

$= \int_0^1 (t,t) \cdot (1,1)dt$

$= \int_0^1 2tdt = t^2|_0^1 = 1$

Closed curve example

Suppose $\mathbf{x} : [0, 2\pi] \to \mathbf{R}^n$, $\mathbf{x}(t) = (cos(t), sin(t))$

Notation 1: Work definition

$\int_C F \cdot ds = \int_0^{2\pi} F(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt$

$= \int_0^{2\pi} F(cos(t), sin(t)) \cdot (-sin(t), cos(t))dt$

$= \int_0^{2\pi} (cos(t), sin(t)) \cdot (-sin(t), cos(t))dt$

$= \int_0^{2\pi} [-cos(t)sin(t) + sin(t)cos(t)]dt$

$= \int_0^{2\pi} 0 \ dt = 0$

Notation 2: differential form

$\int_C F \cdot ds = \int_C F \cdot (dx, dy)$

$= \int_C (x, y) \cdot (dx, dy)$

$= \int_C [xdx + ydy]$

$= \int_0^{2\pi} [cos(t)(-sin(t))dt + sin(t)cos(t)dt]$

$= \int_0^{2\pi} 0 \ dt = 0$

Notation 3: Tangent vector (circulation)

Circulation = 0. Thus $\int_C F \cdot ds = 0$

Method 2: 6.2 Green's Theorem

Let $F = (M, N) = (x, y)$

$\int_C F \cdot ds = \int_C Mdx + Ndy = \int\int_D (\nabla \times f) \cdot \mathbf{k}\, dA$

$= \int\int_D [\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}] dA = \int\int_D [\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}] dA = \int\int_D 0 dA$

Use chapter 5 methods to evaluate double integral.

Note this method is algebraically different than using the definition of vector line integrals. Thus if one method is algebraically difficult, try the other method.

Method 3: 6.3 Claim F has path independent line integrals

Claim F is a gradient field

submethod 1:
$F(x, y) = (x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = \nabla f = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$.

I.e, $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

$\int_C F \cdot ds = 0$ since C is a closed curve.

submethod 2: Claim $(\nabla \times F)(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^n$

Since $n = 2$, $(\nabla \times F)(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^n$ iff $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, where $F = (M, N)$.

$\frac{\partial y}{\partial x} = 0 = \frac{\partial x}{\partial y}$

$\int_C F \cdot ds = 0$ since C is a closed curve.