Scalar Line Integrals:

Let $\mathbf{x} : [a,b] \to \mathbf{R}^n$ be a C^1 path. $f : \mathbf{R}^n \to R$, a scalar field.

$\Delta s_k =$ length of kth segment of path
$= \int_{t_{k-1}}^{t_k} ||\mathbf{x}'(t)||dt = ||\mathbf{x}'(t_k^{**})||(t_k - t_{k-1}) = ||\mathbf{x}'(t_k^{**})||\Delta t_k$
for some $t_k^{**} \in [t_{k-1}, t_k]$

$\int_{\mathbf{x}} f\ ds \sim \Sigma_{i=1}^n f(\mathbf{x}(t_k^*))\Delta s_k = \Sigma_{i=1}^n f(\mathbf{x}(t_k^*))||\mathbf{x}'(t_k^{**})||\Delta t_k$

Thus $\int_{\mathbf{x}} f\ ds = \int_a^b f(\mathbf{x}(t))||\mathbf{x}'(t)||dt$

Vector Line integrals:

Let $\mathbf{x} : [a,b] \to \mathbf{R}^n$ be a C^1 path. $F : \mathbf{R}^n \to R^n$, a vector field.

$\mathbf{x}'(t_k^*) \sim \frac{\Delta \mathbf{x}_k}{\Delta t_k}$

$\int_{\mathbf{x}} F \cdot ds \sim \Sigma_{i=1}^n F(\mathbf{x}(t_k^*)) \cdot \Delta \mathbf{x}_k = \Sigma_{i=1}^n F(\mathbf{x}(t_k^*)) \cdot \mathbf{x}'(t_k^*)\Delta t_k$

Thus $\int_{\mathbf{x}} F \cdot ds = \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t)dt$

Other Formulations of Vector Line integrals:

The tangent vector to $\mathbf{x}$ at t is $T(t) = \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}$

$\int_{\mathbf{x}} F \cdot ds = \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt = \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \frac{||\mathbf{x}'(t)||}{||\mathbf{x}'(t)||} dt =$
$\int_a^b F(\mathbf{x}(t)) \cdot T(t) ||\mathbf{x}'(t)|| dt = \int_{\mathbf{x}} F(\mathbf{x}(t)) \cdot T(t) ds$

Note $\int_a^b F(\mathbf{x}(t)) \cdot T(t) ds$ is a scalar line integral of the scalar field $F \cdot T : \mathbf{R}^n \to \mathbf{R}$ over the path $\mathbf{x}$.

Note: $F \cdot T$ is the tangential component of F along the path $\mathbf{x}$.

Another notation (differential form):

For simplicity, we will work in $\mathbf{R}^2$, but the following generalizes to any dimension.

Let $\mathbf{x}(t) = (x(t), y(t))$. Let $F(x, y) = (M(x, y), N(x, y))$

$x = x(t), y = y(t)$. Also $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$

$$\begin{aligned}
\int_{\mathbf{x}} F \cdot ds &= \int_a^b F(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt \\
&= \int_a^b (M(x, y), N(x, y)) \cdot (x'(t), y'(t)) dt \\
&= \int_a^b (M(x, y), N(x, y)) \cdot (x'(t) dt, y'(t) dt) \\
&= \int_{\mathbf{x}} (M(x, y), N(x, y)) \cdot (dx, dy) \\
&= \int_{\mathbf{x}} M(x, y) dx + N(x, y) dy
\end{aligned}$$

Definitions:

A *curve* is the image of piecewise C^1 path $\mathbf{x} : [a, b] \to \mathbf{R}^n$.

A curve is *simple* if it has no self-intersections; that is, $\mathbf{x}$ is 1:1 on the open interval (a, b)

A path is *closed* if $\mathbf{x}(a) = \mathbf{x}(b)$

A curve is *closed* if there exists a parametrization of the curve such that $\mathbf{x}(a) = \mathbf{x}(b)$

$\int_{\mathbf{x}} F \cdot ds$ is called the *circulation* of f along $\mathbf{x}$ if $\mathbf{x}$ is a closed path.

A *parametrization* of a curve C is a path whose image is C. Normally we will require a parametrization of a curve to be 1:1 where possible.

A piecewise C^1 path $\mathbf{y} : [c, d] \to \mathbf{R}^n$ is a *reparametrization* of a piecewise C^1 path $\mathbf{x} : [a, b] \to \mathbf{R}^n$ if there exists a bijective C^1 function $u : [c, d] \to [a, b]$ where the inverse of u is also C^1 and $\mathbf{y} = \mathbf{x} \circ u$ (i.e., $\mathbf{y}(t) = \mathbf{x}(u(t))$).

Note that either
1.) $u(a) = c$ and $u(b) = d$. In this case, we say that $\mathbf{y}$ (and u are orientation-preserving OR

2.) $u(a) = d$ and $u(b) = c$. In this case, we say that $\mathbf{y}$ (and u are orientation-reversing.

Given piecewise C^1 path, $\mathbf{x} : [a, b] \to \mathbf{R}^n$, the *opposite path* is $\mathbf{x}_{opp} : [a, b] \to \mathbf{R}^n$ $\mathbf{x}_{opp} = \mathbf{x}(a + b - t)$

That is $\mathbf{x}_{opp}$ is an orientation-reversing reparametrization of $\mathbf{x}$ where $u[a, b] \to [a, b]$, $u(t) = a + b - t$.

Thm: Let $\mathbf{x} : [a, b] \to \mathbf{R}^n$ be a piecewise C^1 path and let $\mathbf{y} : [c, d] \to \mathbf{R}^n$ be a reparametrization of $\mathbf{x}$. Then

if $f : \mathbf{R}^n \to \mathbf{R}$ is continuous, then $\int_{\mathbf{y}} f \; ds = \int_{\mathbf{x}} f \; ds$

if $F : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, then

$$\int_{\mathbf{y}} F \cdot ds = \int_{\mathbf{x}} F \cdot ds \text{ if } \mathbf{y} \text{ is orientation-preserving.}$$

$$\int_{\mathbf{y}} F \cdot ds = -\int_{\mathbf{x}} F \cdot ds \text{ if } \mathbf{y} \text{ is orientation-reversing.}$$