A *scalar field* on $\mathbf{R}^n$ is a function $f : \mathbf{R}^n \to \mathbf{R}$.

Visualized by drawing graph in $\mathbf{R}^n \times \mathbf{R}$ or
by drawing level sets in domain $\mathbf{R}^n$.

A *vector field* on $\mathbf{R}^n$ is a function $F : \mathbf{R}^n \to \mathbf{R}^n$.

Visualized by drawing vectors in domain $\mathbf{R}^n$.

Vector Fields can represent many things

Example 1: multiple paths.

Vectors represent velocity (tangent) vectors of paths.

Defn: A *flow line* of a vector field $F : \mathbf{R}^n \to \mathbf{R}^n$ is a differentiable path $\mathbf{x} : I \subset \mathbf{R} \to \mathbf{R}^n$ such that

$$\mathbf{x}'(t) = F(\mathbf{x}(t))$$

That is that tangent vector to $\mathbf{x}$ at time t is $\mathbf{x}'(t) = F(\mathbf{x}(t))$.

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Ex 8 (p. 212): $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(x, y) = (-y, x)$

To find flow line(s) need $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ such that $\mathbf{x}'(t) = F(\mathbf{x}(t))$

$$\text{I.e, need } (x'(t), y'(t)) = F(x(t), y(t))$$

$$\text{I.e, need } (x'(t), y'(t)) = (-y(t), x(t))$$

That is need to solve $x'(t) = -y(t)$ and $y'(t) = x(t)$.

Solution: $\mathbf{x}(t) = (acost - bsint, asint + bcost)$

Check: $\mathbf{x}'(t) = (-asint - bcost, acost - bsint)$

$F(\mathbf{x}(t)) =$
$F(acost - bsint, asint + bcost) = (-asint - bcost, acost - bsint)$

Example 2: A vector field can represent a gradient field.

Vectors = ∇f, for some scalar field f.

Defn: A *gradient field* on $\mathbf{R}^n$ is a vector field $F : \mathbf{R}^n \to \mathbf{R}^n$ such that $F(x, y) = \nabla f$, for some scalar field $f : \mathbf{R}^n \to \mathbf{R}$

f is called the *potential function* for F.

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Ex: $f : \mathbf{R}^2 \to \mathbf{R}, \qquad f(x, y) = x^2 + 3y^2, \qquad \nabla f = (2x, 6y).$

$F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(x, y) = (2x, 6y)$ is a gradient field with potential function f.

Equipotential set of F = level set of f.

Thus the vector $F(x, y)$ is perpendicular to an equipotential set of F = level set of f since $F(x, y) = \nabla f$ points in the direction of steepest ascent for the terrain described by the graph of f in $\mathbf{R}^2 \times \mathbf{R}$.

Magnitude of the vector $F(x, y)$ indicate the steepness of the slope

Calc 1 Review:

Taylor's Thm for $f : \mathbf{R} \to \mathbf{R}$. Suppose $f \in C^k$,

Let $p_k(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k$

Then $f(x) = p_k(x) + R_k(x,a)$ where $lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0$.

Prop 1.2: If $f^{(k+1)}$ exists, then there exists c between a and x such that $R_k(x,a) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$

Estimate $ln(2)$ using degree 3 Taylor polynomial for $f(x) = ln(x)$ about $a = 1$

$f(x) = ln(x)$ $\quad f'(x) = x^{-1}$ $\quad f''(x) = -x^{-2}$ $\quad f'''(x) = 2x^{-3}$

$p_3(x) =$

Thus $ln(2) \sim p_3(2) =$

$ln(x) = p_3(x) + R_3(x-1)$

$f^{(4)}(x) = -6x^{-4} = -\frac{6}{x^4}$

$R_3(x-1) = \frac{f^{(4)}(c)}{(4)!}(x-1)^4 =$

where c is btwn 1 and 2.

$ln(2) = p_3(2) + R_3(2,1) =$ where $c \in (1,2)$

Multivariable version:

Taylor's Thm for $f : \mathbf{R} \rightarrow \mathbf{R}$. Suppose $f \in C^k$,

Let $p_k(\mathbf{x}) = f(\mathbf{a}) + \Sigma_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + ...$

$$+\frac{1}{k!}\Sigma_{i_1,...,i_k=1}^{n} f_{i_1...i_k}(\mathbf{a})(x_{i_1} - a_{i_1})...(x_{i_k} - a_{i_k})$$

Then $f(\mathbf{x}) = p_k(\mathbf{x}) + R_k(\mathbf{x}, \mathbf{a})$ where $lim_{x \rightarrow a} \frac{R_k(x,a)}{||x-a||^k} = 0$.

If $f : \mathbf{R}^n \rightarrow \mathbf{R} \in C^2$, then

$R_1(\mathbf{x}, \mathbf{a}) = \frac{1}{2}\Sigma_{i,j=1}^{n} f_{x_i x_j}(\mathbf{c})(x_i - a_i)(x_j - a_j)$

for some $\mathbf{c}$ on the line segment joining $\mathbf{a}$ and $\mathbf{x}$.

If $f : \mathbf{R}^n \rightarrow \mathbf{R} \in C^{k+1}$, then

$(\mathrm{x}_{i_{k+1}} - a_{i_{k+1}})$

$R_k(\mathbf{x}, \mathbf{a}) = \frac{1}{(k+1)!}\Sigma_{i_1,...,i_{k+1}=1}^{n} f_{x_{i_1}...x_{i_{k+1}}}(\mathbf{c})(x_{i_1} - a_{i_1})...(x_{i_{k+1}} - a_{i_{k+1}})$

for some $\mathbf{c}$ on the line segment joining $\mathbf{a}$ and $\mathbf{x}$.