Let $p : \mathbf{R}^2 \to \mathbf{R}^2$, $p(r,\theta) = (rcos\theta, rsin\theta)$.

Then $Dp(r,\theta) = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix}$

Let $f : \mathbf{R}^2 \to \mathbf{R}$, $f(x,y) = x^2 + y^2 + 3y$

$$(f \circ p)(r,\theta) = f(p(r,\theta)) = f(rcos\theta, rsin\theta)$$
$$= r^2cos^2\theta + r^2sin^2\theta + 3rsin\theta = r^2 + 3rsin\theta$$

Note we used the function p to convert $x^2 + y^2 + 3y$ into polar coordinates.

Use the chain rule to calculate $D(f \circ p)(r,\theta)$:

$f(x,y) = x^2 + y^2 + 3y$. Thus, $Df(x,y) = \begin{pmatrix} 2x & 2y+3 \end{pmatrix}$

$$D(f \circ p)(r,\theta) = (Df)(p(r,\theta))Dp(r,\theta)$$
$$= \begin{pmatrix} 2rcos\theta & 2rsin\theta + 3 \end{pmatrix} \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} =$$

$$\begin{pmatrix} 2rcos^2\theta + 2rsin^2\theta + 3sin\theta & -2r^2sin\theta cos\theta + 2r^2sin\theta cos\theta + 3rcos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 2r + 3sin\theta & 3rcos\theta \end{pmatrix}$$

$$D(f \circ p)(r,\theta) = (Df)(p(r,\theta))Dp(r,\theta)$$
$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x}cos\theta + \frac{\partial f}{\partial y}sin\theta & -\frac{\partial f}{\partial x}rsin\theta + \frac{\partial f}{\partial y}rcos\theta \end{pmatrix}$$

$$\frac{\partial}{\partial r} = cos\theta\frac{\partial}{\partial x} + sin\theta\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = -rsin\theta\frac{\partial}{\partial x} + rcos\theta\frac{\partial}{\partial y}$$

$$\begin{pmatrix} cos\theta & sin\theta & \frac{\partial}{\partial r} \\ -rsin\theta & rcos\theta & \frac{\partial}{\partial \theta} \end{pmatrix} \rightarrow \begin{pmatrix} rcos\theta sin\theta & rsin^2\theta & rsin\theta\frac{\partial}{\partial r} \\ -rcos\theta sin\theta & rcos^2\theta & cos\theta\frac{\partial}{\partial \theta} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} rcos\theta sin\theta & rsin^2\theta & rsin\theta\frac{\partial}{\partial r} \\ 0 & r & cos\theta\frac{\partial}{\partial \theta} + rsin\theta\frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} cos\theta & sin\theta & \frac{\partial}{\partial r} \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} cos\theta & 0 & \frac{\partial}{\partial r} - sin\theta[\frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r}] \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} cos\theta & 0 & \frac{\partial}{\partial r} - sin\theta\frac{cos\theta}{r}\frac{\partial}{\partial \theta} - sin^2\theta\frac{\partial}{\partial r} \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix}$$

$$\rightarrow\begin{pmatrix} cos\theta & 0 & -sin\theta\frac{cos\theta}{r}\frac{\partial}{\partial \theta} + (1 - sin^2\theta)\frac{\partial}{\partial r} \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix}\rightarrow$$

$$\begin{pmatrix} cos\theta & 0 & -sin\theta\frac{cos\theta}{r}\frac{\partial}{\partial \theta} + cos^2\theta\frac{\partial}{\partial r} \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & \frac{-sin\theta}{r}\frac{\partial}{\partial \theta} + cos\theta\frac{\partial}{\partial r} \\ 0 & 1 & \frac{cos\theta}{r}\frac{\partial}{\partial \theta} + sin\theta\frac{\partial}{\partial r} \end{pmatrix} \blacksquare$$

$$\frac{\partial}{\partial x} = cos\theta\frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = sin\theta\frac{\partial}{\partial r} + \frac{cos\theta}{r}\frac{\partial}{\partial \theta}$$