Calculus I review:

Suppose $f : \mathbf{R} \to \mathbf{R}$.

Recall the tangent line to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

Thus $y = f(a) + f'(a)(x - a)$ is the best linear approximation of f near $x = a$.

Ex: Find the best linear approximation for $f(x) = 2x + 5$.

Answer:

Note slope $= f'(x) = 2$.

Ex: Find the best linear approximation for $h(x) = x^2$ at $x = 3$.

$h'(x) = 2x$.
Thus slope of tangent line at $x = 3$ is $h'(3) = 2(3) = 6$.

Hence $\frac{y-9}{x-3} = 6$

Equation of tangent line: $y = 9 + 6(x - 3)$

Estimate $h(3.1)$:

The gradient of $f : R^n \to R^1$ is denoted by

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), ..., \frac{\partial f}{\partial x_n}(\mathbf{a})\right)$$

Defn: The **Jacobian matrix of** $f : R^n \to R^m$ **at** $\mathbf{a}$ is

$$Df(\mathbf{a}) = \left[\, \frac{\partial f_i}{\partial x_j}(\mathbf{a}) \,\right]_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

Thm: If f is differentiable at $\mathbf{a}$, then

1.) f is continuous at $\mathbf{a}$.

2.) $\frac{\partial f_i}{\partial x_j}$ exists at $\mathbf{a}$ for all i, j.

3.) The derivative of f at $\mathbf{a} = Df(\mathbf{a})$
$=$ the Jacobian matrix of f at $\mathbf{a}$.

Thm: Let $f : \mathbf{R}^n \to \mathbf{R}^m$, $f = (f_1, ..., f_m)$. If $\frac{\partial f_i}{\partial x_j}$ exists and are continuous in a neighborhood of $\mathbf{a}$ for all i, j, then f is differentiable at $\mathbf{a}$