1.1 Vectors:

Let $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

If $\mathbf{v}$ = velocity in m/sec of an object, then the object is moving east at a rate of 1 m/sec and north at a rate of 2m/sec

Speed of the object =

A vector can be described by its Euclidean coordinates OR by its length and direction.

Let $\mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Then $\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} =$

$\mathbf{v} - \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} =$

2.1 Let $f : X \to Y$ where $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$

Graph of $f = \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in X\} \subset \mathbf{R}^n \times \mathbf{R}^m$

Domain of $f = X$, Codomain of $f = Y$

Range of f = Image of $f = f(X)$
$= \{y \in Y \mid$ there exists $x \in X$ such that $f(x) = y\}$.

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f is a function if for all x in domain of f, $f(x)$ has a unique value.
I.e, for all $x, y \in X$, if $x = y$, then $f(x) = f(y)$
and for all $x \in X$, $f(x)$ is defined.

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f is 1:1 if $f(x) = f(y)$ implies $x = y$.

f gives a one-to-one correspondence between X and $f(X)$.

Given $b \in Y$, $f(x) = b$ has at most one solution

Side-note: $f(x) = b$ has exactly one solution if $b \in f(X)$.
. Side-note: $f(x) = b$ has no solution if $b \notin f(X)$.

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f is onto if $f(X) = Y$ (i.e., image of f = codomain of f).

Given $b \in Y$, $f(x) = b$ has at least one solution.

Ex 1: $f : \mathbf{R}^n \to \mathbf{R}$, $f(\mathbf{x}) = ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

Domain = Codomain = Image =

Is f 1:1?

Proof:

Is f onto?

Proof:

Alternate Proof:

Ex 2: $g(x, y) = (x^2 y, x^4 - y, x^6)$

Domain = Codomain =

Is g 1:1?

Proof:

Is g onto?

Proof:

Ex 3: $h(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

I.e, $h(\mathbf{x}) = (x + 2y + 3z, 4x + 5y + 6z)$.

Domain $=$ Codomain $=$ Image $=$

Is h onto? Is h 1:1?

How many solutions does $h(\mathbf{x}) = \mathbf{b}$ have?

I.e., how many solutions does $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ have?

I.e, how many solutions does the following system of equations have:

$$x + 2y + 3z = b_1,$$

$$4x + 5y + 6z = b_2.$$

Does $\begin{pmatrix} 1 \\ 4 \end{pmatrix} x + \begin{pmatrix} 2 \\ 5 \end{pmatrix} y + \begin{pmatrix} 3 \\ 6 \end{pmatrix} z$ span all of $\mathbf{R}^2$?

Is $\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}\}$ linearly independent?

Definitions:

If the codomain of f is $\mathbf{R}$ (i.e., $f : X \to \mathbf{R}$), we say that f is *real-valued* or *scalar valued.*

Suppose $f : X \subset \mathbf{R}^2 \to \mathbf{R}$ and c is a constant scalar.

The *level curve at height* c *of* f is the curve in $\mathbf{R}^2$ defined by $f(x, y) = c$. That is,

$$\text{the level curve at height } c \text{ of } f = \{(x, y) \in \mathbf{R}^2 \mid f(x, y) = c\}.$$

The *contour curve at height* c *of* f is the curve in $\mathbf{R}^3$ defined by the two equations, $z = f(x, y), z = c$. That is,

$$\begin{aligned} &\text{the contour curve at height } c \text{ of } f = \\ &\qquad = \{(x, y, z) \in \mathbf{R}^2 \mid z = f(x, y) = c\} \\ &\qquad = \{(x, y, f(x, y)) \in \mathbf{R}^3 \mid f(x, y) = c\}. \end{aligned}$$

$$\begin{aligned} \text{Recall the graph of } f &= \{(x, y, z) \mid z = f(x, y)\} \\ &= \{(x, y, f(x, y)) \mid (x, y) \in X\} \subset \mathbf{R}^2 \times \mathbf{R} \end{aligned}$$

The *section of the graph of* f *by the plane* $x = c$ is the set of points in $\mathbf{R}^3$ defined by the two equations, $z = f(x, y), x = c$. That is,

$$\begin{aligned} \text{the section by } x = c \text{ is } &\{(x, y, z) \in \mathbf{R}^2 \mid z = f(x, y), x = c\} \\ &= \{(c, y, f(c, y)) \in \mathbf{R}^3 \mid (c, y) \in X\}. \end{aligned}$$

$$\begin{aligned} \text{The section by } y = c \text{ is } &\{(x, y, z) \in \mathbf{R}^2 \mid z = f(x, y), y = c\} \\ &= \{(x, c, f(x, c)) \in \mathbf{R}^3 \mid (x, c) \in X\}. \end{aligned}$$