Section 1.6: $\mathbf{R}^n$

Thm (Cauchy-Schwarz Inequality): Let $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^n$. Then

$$\mathbf{u} \cdot \mathbf{v} \;\leq\; ||\mathbf{u}||\;||\mathbf{v}||$$

Recall in $\mathbf{R}^2$ and in $\mathbf{R}^3$, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\;||\mathbf{v}||cos\theta$

Thm (Triangle Inequality): Let $\mathbf{u}$ and $\mathbf{v} \in \mathbf{R}^n$. Then

$$||\mathbf{u}+\mathbf{v}|| \;\leq\; ||\mathbf{u}|| + ||\mathbf{v}||$$

A matrix is a vector of vectors

Example of a 2×3 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Operations on Matrices

$A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$.

Defn: Two matrices A and B are equal, if they have the same size and $a_{ij} = b_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$

Defn: $A + B = (a_{ij} + b_{ij})$.

Defn: $cA = (ca_{ij})$.

Defn: $-B = (-1)B$.

Defn: $A - B = A + (-B)$.

Defn: The zero matrix $= \mathit{0} = (a_{ij})$ where $a_{ij} = 0$ for all i, j.

Defn: The identity matrix $= I = (a_{ij})$ where $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1$ for all i and I is a square matrix.

The transpose of the $m \times n$ matrix $A = A^T = (a_{ji})$.

Suppose A is an $n \times r$ matrix, B is an $r \times n$. $AB = C$ where

$$c_{ij} = row(i)\ of\ A \cdot column(j)\ of\ B$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ir}b_{rj}.$$

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 10 & 1 \\ 0 & 4 \end{bmatrix} =$

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jth column of $AB = A[j$th column of $B]$

In other words $AB = A[b_1 ... b_n] = [Ab_1 ... Ab_n]$

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ith row of $AB = [i$th row of $A]B$

In other words $AB = \begin{bmatrix} a(1) \\ . \\ . \\ . \\ a(n) \end{bmatrix} B \quad = \begin{bmatrix} a(1)B \\ . \\ . \\ . \\ a(n)B \end{bmatrix}$

Thm (Properties of matrix arithmetic) Let A, B, C be matrices. Let a, b be scalars. Assuming that the following operations are defined, then

a.) $A + B = B + A$

b.) $A + (B + C) = (A + B) + C$

c.) $A + 0 = A$

d.) $A + (-A) = 0$

e.) $A(BC) = (AB)C$

f.) $AI = A, IB = B$

g.) $A(B + C) = AB + AC,$
. $(B + C)A = BA + CA$

h.) $a(B + C) = aB + aC$

i.) $(a + b)C = aC + bC$

j.) $(ab)C = a(bC)$

k.) $a(AB) = (aA)B = A(aB)$

l.) $1A = A$

Defn.) $-A = -1A$

Cor.) $A0 = 0, 0B = 0$

Cor.) $a0 = 0$

a.) $(A^T)^T = A$

b.) $(A + B)^T = A^T + B^T$

c.) $(kA)^T = kA^T$

d.) $(AB)^T = B^T A^T$

Note

$$AB \neq BA$$

It is also possible that $AB = AC$, but $B \neq C$.

In particular it is possible for $AB = 0$, but $A \neq 0$ AND $B \neq 0$

Defn: A is invertible if there exists a matrix B such that $AB = BA = I$, and B is called the inverse of A. If the inverse of A does not exist, then A is said to be singular.

Ex: $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} =$

Ex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} =$

Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} =$

Note that if A is invertible, then A is a square matrix.

Thm: If A is invertible, then its inverse is unique.

Proof: Suppose $AB = I$ and $CA = I$.

Then, $B = IB = CAB = CI = C$.

Defn: A **linear transformation** is a function $T : \mathbf{R}^n \to \mathbf{R}^m$ that satisfies the following two conditions. For each $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ and scalar a,

i.) $T(a\mathbf{u}) = aT(\mathbf{u})$

ii.) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Thm: Let $T : V \to W$ be a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$

Pf: $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$

Thm: Let A be an $m \times n$ matrix. Then the function

$$T : \mathbf{R}^n \to \mathbf{R}^m$$
$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

Prove that $T : R^3 \to R^2$, $T(x, y, z) = (4x - 5y, 8)$ is NOT a linear transformation.

Examples of linear transformations:

Ex: If $T : \mathbf{R}^2 \to \mathbf{R}$ is a linear transformation and $T(1,0) = 3$, $T(0,1) = 4$, then

$T(x, y) =$

Thm 4.3.3: If $T : R^n \to R^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ where $A = [T(\mathbf{e_1})...T(\mathbf{e_n})]$.

Proof: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1\mathbf{e_1} + ... + x_n\mathbf{e_n}$.

$$\begin{aligned} \text{Thus } T(\mathbf{x}) &= T(x_1\mathbf{e_1} + ... + x_n\mathbf{e_n}) \\ &= x_1T(\mathbf{e_1}) + ... + x_nT(\mathbf{e_n}) \\ &= [T(\mathbf{e_1})...T(\mathbf{e_n})] \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} \end{aligned}$$