Exam 2 March 31, 2011 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

[10] 1a.) Use a linear approximation (or differentials) to estimate $ln(0.97)$

Let $f(x) = ln(x)$. Note 0.97 is close to 1 and $f(1) = ln(1) = 0$

$f'(x) = \frac{1}{x}$. Hence the slope of the tangent line to f at $x = 1$ is $\frac{dy}{dx} = f'(1) = 1$.

Since $\Delta x = dx = 0.97 - 1 = -0.03$, $dy = f'(1)dx = 1(-0.03) = -0.03$.

Thus $ln(0.97) = f(1) + \Delta y = 0 + \Delta y \sim dy = -0.03$

Alternate method: Since $f(1) = 0$ and $f'(1) = 1$, the equation of the tangent line to $f(x) = ln(x)$ at $x = 1$ is $y = x - 1$. Thus the tangent line $L(x) = x - 1$ is the linear approximation to $f(x) = ln(x)$ near $x = 1$. Since 0.97 is close to 1, $f(x) \sim L(x) = 0.97 - 1 = -0.3$.

Sidenote: In this case, the answer is a good approximation ($ln(0.97) = -0.0304592...$), but whether or not a tangent line is a good approximation of a function near the point of tangency depends on the function.

Answer 1a.) $\underline{-0.03}$

[2] 1b.) Is the answer to 1a an over-estimate or an under-estimate? $\underline{over - estimate}$

$f'(x) = x^{-1}$, $f''(x) = -x^{-2} < 0$. Hence f is concave down and the answer is an over-estimate.

[2] 1c.) For the problem in 1a,

$dx = \underline{-0.03}$, $dy = \underline{-0.03}$, $\Delta x = \underline{-0.03}$, $\Delta y = \underline{ln(0.97)}$.

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[2] 2a.) The linearization of $f(x) = sin(x)$ at $x = 0$ is $\underline{y = x}$

[2] 2b.) The linearization of $f(x) = cos(x)$ at $x = 0$ is $\underline{y = 1}$

[2] 2c.) The linearization of $f(x) = sin(x)$ at $x = \frac{\pi}{2}$ is $\underline{y = 1}$

[2] 2d.) The linearization of $f(x) = 2x + 1$ at $x = 0$ is $\underline{y = 2x + 1}$

[2] 2e.) Use the above linearizations to estimate the following:

$sin(0.1) \sim \underline{0.1}$, $sin(\frac{3}{2}) \sim \underline{1}$

[10] 3.) Recall that radioactive substances decay at a rate proportional to the remaining mass. The half-life of polonium-218 is 3 minutes. Suppose a sample originally has a mass of 400g. SIMPLIFY your answers to the following:

a.) A formula for the mass remaining after t minutes is $\underline{\ 400(2^{-t/3})\ }$

b.) The mass remaining after 6 minutes is $\underline{\ 100\ }$

c.) When is the mass reduced to 10g? $\underline{\ log_2(64000)\ }$

Answer:

b.) If half life = 3 minutes and if $m(0) = 400$, then $m(3) = 200$ and $m(6) = 100$

a.) radioactive substances decay at a rate proportional to the remaining mass: $\frac{dm}{dt} = km$

Note $m(t) = m(0)e^{kt}$ is a solution to $\frac{dm}{dt} = km$

From ch 5: $\int \frac{dm}{m} = \int kdt$. hence $ln|m| = kt + C$. Thus $|m| = e^{kt+C} = e^{kt}e^C$. Since mass is always positive, we obtain $m(t) = m(0)e^{kt}$

$400e^{3t} = 200$. Thus $e^{3k} = \frac{1}{2}$. $k = -\frac{ln(2)}{3}$

$m(t) = 400e^{-\frac{ln(2)}{3}t} = 400e^{ln(2)\frac{-t}{3}} = 400(2^{-t/3})$

b.) $m(6) = 400(2^{-6/3}) = 400(2^{-2}) = 400/4 = 100$

c.) $10 = 400(2^{-t/3})$

$\frac{1}{40} = 2^{-t/3}$

Taking the recipricol of both sides: $40 = 2^{t/3}$

$log_2(40) = t/3$

$t = 3log_2(40) = log_2(40^3) = log_2(64000)$

Answer) ____

[15] 4.) Find the derivative of $x\ ln(\sqrt{3sin(x^2) - e^x + 1})$.
Circle your answer. You do NOT need to simplify.

$[x\ cos(ln(\sqrt{3e^x - x^2 + 1}))]' =$

$ln(\sqrt{3sin(x^2) - e^x + 1}) + x(\frac{1}{\sqrt{3sin(x^2)-e^x+1}})(\frac{1}{2})(3sin(x^2) - e^x + 1)^{-\frac{1}{2}}(6xcos(x^2) - e^x)$

[15] 5.) A plan flies horizontally at an altitude of 10km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$ (with respect to the tracking telescope, after it has passed over the tracking telescope), this angle is decreasing at a rate of $\pi/4$ rad/min. How fast is the plane traveling at that time. (Hint: you can use a right triangle).

$tan(\theta) = \frac{10}{x}$. Hence $\frac{sin\theta}{cos\theta} = \frac{10}{x}$.   Note when $\theta = \pi/3$, $\theta' = -\pi/4$

method 1: $xsin(\theta) = 10cos(\theta)$

$x'sin(\theta) + xcos(\theta)\theta' = -10sin(\theta)\theta'$

when $\theta = \pi/3$, $\theta' = -\pi/4$. $tan(\pi/3) = \frac{10}{x}$ implies $x = \frac{10}{tan(\pi/3)} = \frac{10}{\sqrt{3}}$.

$x'sin(\pi/3) + \frac{10}{\sqrt{3}}cos(\pi/3)(-\frac{\pi}{4}) = -10sin(\pi/3)(-\frac{\pi}{4})$.

$x'\frac{\sqrt{3}}{2} + \frac{10}{\sqrt{3}}(\frac{1}{2})(-\frac{\pi}{4}) = -10(\frac{\sqrt{3}}{2})(-\frac{\pi}{4})$.

$x'\sqrt{3} + \frac{10}{\sqrt{3}}(-\frac{\pi}{4}) = -10\sqrt{3}(-\frac{\pi}{4})$.

$x'\sqrt{3} = -10\sqrt{3}(-\frac{\pi}{4}) - \frac{10}{\sqrt{3}}(-\frac{\pi}{4}). = 10\sqrt{3}(\frac{\pi}{4}) + \frac{10}{\sqrt{3}}(\frac{\pi}{4})$.

$x' = 10(\frac{\pi}{4}) + \frac{10}{3}(\frac{\pi}{4}) = \frac{40}{3}(\frac{\pi}{4}) = \frac{10\pi}{3}$.

method 2: $x = 10\frac{cos(\theta)}{sin(\theta)}$

$x' = 10(\frac{-sin(\theta)\theta' sin(\theta) - cos(\theta)cos(\theta)\theta'}{sin^2\theta}) = -10\theta'(\frac{sin^2(\theta)+cos^2(\theta)}{sin^2\theta}) = \frac{-10\theta'}{sin^2\theta}$

when $\theta = \pi/3$, $\theta' = -\pi/4$, $x' = \frac{10\pi/4}{sin^2(\pi/3)} = \frac{10\pi/4}{(\frac{\sqrt{3}}{2})^2} = \frac{10\pi/4}{3/4} = \frac{10\pi}{3}$

Answer) $\frac{10\pi}{3}km/min$

6.) Find the following for $f(x) = x^3 - 8x^2 + 16x = x(x-4)^2$ (if they exist; if they don't exist, state so). Use this information to graph f.

Note $f'(x) = 3x^2 - 16x + 16 = (x-4)(3x-4)$ and $f''(x) = 6x - 16$

[1.5] 6a.) critical numbers: $\underline{\frac{4}{3}, 4}$

[1.5] 6b.) local maximum(s) occur at $x = \underline{\ \frac{4}{3}\ }$

[1.5] 6c.) local minimum(s) occur at $x = \underline{\ 4\ }$

[1.5] 6d.) The global maximum of f on the interval [0, 5] is $\underline{\frac{256}{27}}$ and occurs at

$x = \underline{\frac{4}{3}}$

[1.5] 6e.) The global minimum of f on the interval [0, 5] is $\underline{0}$ and occurs at

$x = \underline{0, 4}$

[1.5] 6f.) Inflection point(s) occur at $x = \underline{\frac{8}{3}}$

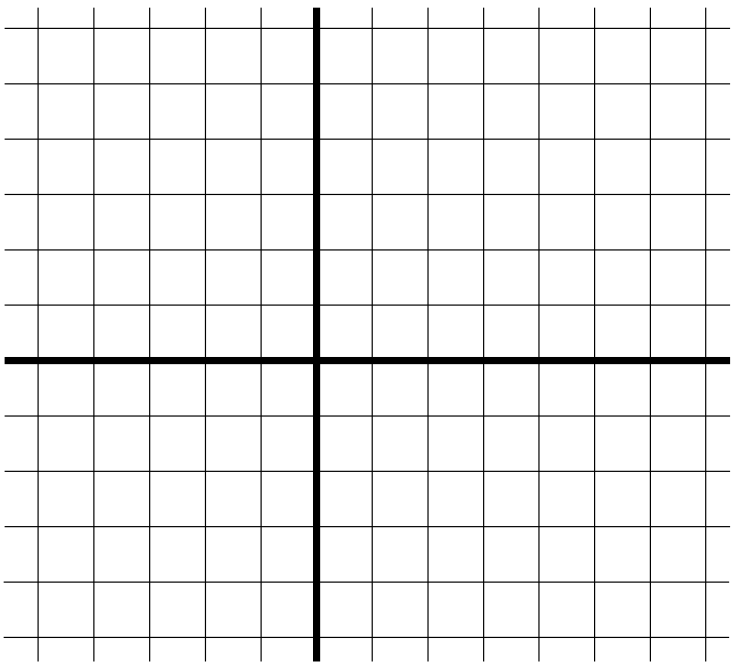
[1.5] 6g.) f increasing on the intervals $\underline{(-\infty, \frac{4}{3}) \cup (4, \infty)}$

[1.5] 6h.) f decreasing on the intervals $\underline{(\frac{4}{3}, 4)}$

[1.5] 6i.) f is concave up on the intervals $\underline{(\frac{8}{3}, \infty)}$

[1.5] 6j.) f is concave down on the intervals$\underline{(-\infty, \frac{8}{3})}$

[5] 6k.) Graph f



[16] 7.) Circle T for true and F for false. If the statement is false, give a counter-example.

7a.) If $f'(x) > 0$ on an interval, then f is increasing on that interval. T

Counter-example: $\underline{None}$.

7b.) If f is increasing on an interval, then $f'(x) > 0$ on that interval. F

Counter-example: $\underline{f(x) = x^3 \text{ is an increasing function on } (-\infty, \infty), \text{ but } f'(0) = 0}$.

7c.) If $f'(c) = 0$, then f has a local maximum or local minimum at c. F

Counter-example: If $f(x) = x^3$, then $f'(0) = 0$, but $f(0)$ is neither a local maximum nor a local minimum

7d.) If f has a local maximum or local minimum at c, then $f'(c) = 0$. F

Counter-example: $f(x) = |x|$ has a local minimum at $x = 0$, but $f'(0)$ does not exist.

7e.) If f has a local maximum or local minimum at c and if $f'(c)$ exists, then $f'(c) = 0$. T

Counter-example: None

7f.) Suppose f'' is continuous near x and $f'(c) = 0$. If $f''(c) > 0$, then f has a local minimum at c. T

Counter-example: None

7g.) Suppose f'' is continuous near x and $f'(c) = 0$. If f has a local minimum at c, then $f''(c) > 0$. F

Counter-example: $f(x) = x^4$ has a local minimum at $x = 0$, but $f''(0) = 0$.

7h.) If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ at some number c in (a, b). F

Counter-example: $f : (0, 1) \to R, f(x) = x$ does not have an absolute maximum value.