Final Exam PART A, Dec., 2005 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

All problems required on this part of the exam.

1.) Suppose $f(x) = 2x^2 - x + 1$ and $g(x) = x^2 + 2x + 5$.

$2x^2 - x + 1 = x^2 + 2x + 5$

$x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$. Hence $x = -1, 4$

Take $0 \in (-1, 4)$

$x = 0 : 2x^2 - x + 1 = 1$

$x = 0 : x^2 + 2x + 5 = 5$

Hence $x^2 + 2x + 5 > 2x^2 - x + 1$ on (-1, 4)

[3] 1a.) Set up, **but do NOT evaluate**, an integral for the area of the region enclosed by f and g.

height $= x^2 + 2x + 5 - (2x^2 - x + 1) = -x^2 + 3x + 4$, width $= dx$

Area $= \int_{-1}^{4}(-x^2 + 3x + 4)dx$

[4] 1b.) Set up, **but do NOT evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the curves f and g about the line $x = 9$ (hint: use cylindrical shells).

Area $= 2\pi rh$, $h = x^2 + 2x + 5 - (2x^2 - x + 1) = -x^2 + 3x + 4$, $r = 9 - x$,

Area $= 2\pi(9 - x)(-x^2 + 3x + 4)$

width or thickness of cylindrical shell $= dx$.

Volume $= 2\pi \int_{-1}^{4}(9 - x)(-x^2 + 3x + 4)dx$

[4] 1c.) Set up, **but do NOT evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the curves f and g about the line $y = -2$ (hint: use washers).

Area $= \pi(R^2 - r^2)$, $R = x^2 + 2x + 5 - (-2) = x^2 + 2x + 7$, $r = 2x^2 - x + 1 - (-2) = 2x^2 - x + 3$

width or thickness of washer $= dx$.

Volume $= \pi \int_{-1}^{4}[(x^2 + 2x + 7)^2 - (2x^2 - x + 3)^2]dx$

[1] 2.) If $h(x) = x^2$, then the slope of the tangent line at the point (2, 4) is $\underline{4}$

$h'(x) = 2x$, $h'(2) = 2(2) = 4$

[10] 3.) Find the derivative of $g(x) = \sqrt{\frac{ln(sin(x))}{x+1}}$

Answer 3.) $\underline{\frac{1}{2}[\frac{ln(sin(x))}{x+1}]^{\frac{-1}{2}}[\frac{(x+1)(\frac{1}{sin(x)})cos(x)-ln(sin(x))(1)}{(x+1)^2}]}$

4.) Find the following integrals:

Note typo: the bottom should be $\sqrt{sin(x)+4}$

[10] a.) $\int_0^{\frac{\pi}{2}} \frac{cosx}{\sqrt{sin(x)+4}}dx = \underline{2\sqrt{5}-4}$

let $u = sin(x) + 4$

$du = cos(x)dx$

$x = 0 : u = sin(0) + 4 = 0 + 4 = 4$

$x = \frac{\pi}{2} : u = sin(\frac{\pi}{2}) + 4 = 1 + 4 = 5$

$\int_0^{\frac{\pi}{2}} \frac{cosx}{\sqrt{sin(x)+4}}dx = \int_4^5 \frac{du}{u^{\frac{1}{2}}} = \int_4^5 u^{\frac{-1}{2}}du = 2u^{\frac{1}{2}}|_4^5 = 2[\sqrt{5}-\sqrt{4}] = 2[\sqrt{5}-2] = 2\sqrt{5}-4$

[3] b.) $\int \frac{5}{1+x^2}dx = \underline{5tan^{-1}(x) + C}$ (from table)

Note typo: Should be right-handed limit.

[10] 5.) Find the following limit (SHOW ALL STEPS): $lim_{x\to 0^+}(3x+1)^{\frac{1}{x^2}} = \underline{+\infty}$

$lim_{x\to 0^+}(3x+1)^{\frac{1}{x^2}} = lim_{x\to 0^+}e^{ln[(3x+1)^{\frac{1}{x^2}}]}$

$lim_{x\to 0^+}ln[(3x+1)^{\frac{1}{x^2}}] = lim_{x\to 0^+}\frac{1}{x^2}ln(3x+1) = lim_{x\to 0^+}\frac{ln(3x+1)}{x^2}$ ("$\frac{0}{0}$")

$= lim_{x\to 0^+}\frac{\frac{1}{3x+1}(3)}{2x}$ (by l'Hospital's rule)

$= lim_{x\to 0^+}\frac{3}{(3x+1)2x} = +\infty$

$lim_{x\to 0^+}(3x+1)^{\frac{1}{x^2}} = lim_{x\to 0^+}e^{ln[(3x+1)^{\frac{1}{x^2}}]} = e^{lim_{x\to 0^+}ln[(3x+1)^{\frac{1}{x^2}}]} = +\infty$

6.) Find the following for $f(x) = xe^{-2\sqrt{x}}$ (if they exist; if they don't exist, state so). Use this information to graph f.

Note $f'(x) = e^{-2\sqrt{x}}(1 - \sqrt{x})$ and $f''(x) = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}(\sqrt{x} - \frac{3}{2})$ and $lim_{x\to\infty} xe^{-2\sqrt{x}} = 0$

[1] 6a.) critical numbers: $\underline{x = 1}$

[1] 6b.) local maximum(s) occur at $x = \underline{x = 1}$

[1] 6c.) local minimum(s) occur at $x = \underline{none}$

[1] 6d.) The global maximum of f on the interval [0, 5] is $\underline{e^{-2}}$ and occurs at $x = \underline{1}$

[1] 6e.) The global minimum of f on the interval [0, 5] is $\underline{0}$ and occurs at $x = \underline{0}$ (since $f(0) = 0$, $f(1) = e^{-2}$, $f(5) = 5e^{-2\sqrt{5}}$

[1] 6f.) Inflection point(s) occur at $x = \underline{\frac{9}{4}}$

[1] 6g.) f increasing on the intervals $\underline{[0, 1)}$

[1] 6h.) f decreasing on the intervals $\underline{(1, \infty)}$

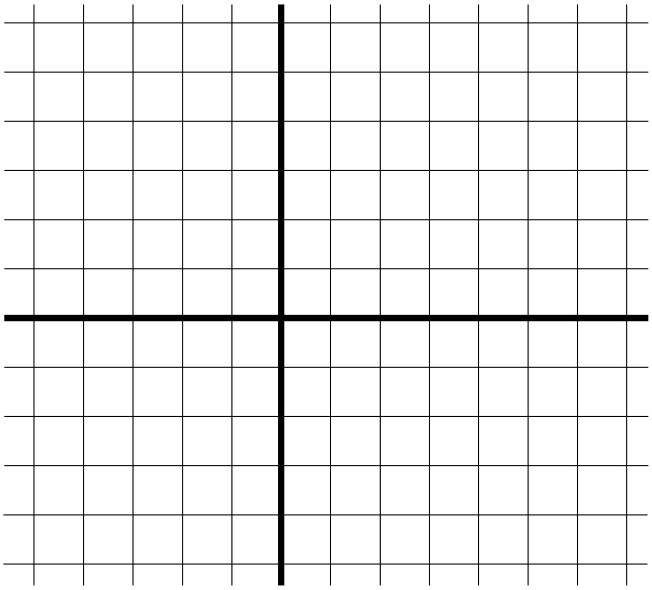
[1] 6i.) f is concave up on the intervals $\underline{[\frac{9}{4}, \infty)}$

[1] 6j.) f is concave down on the intervals$\underline{(0, \frac{9}{4})}$

[1] 6k.) What is the domain of f? $\underline{[0, \infty)}$

[1] 6l.) What is the range of f? $\underline{[0, e^{-2}]}$

[4] 6m.) Graph f

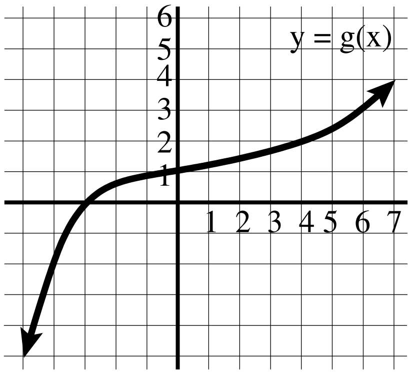


Final Exam PART B, Dec., 2005 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

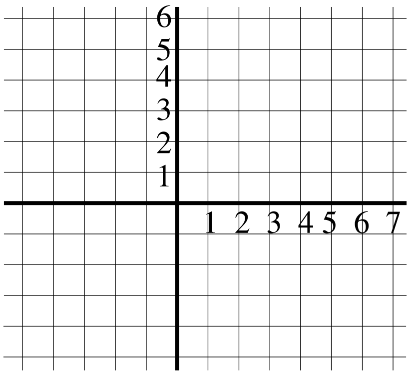
Choose 4 out of the following 5 problems: **Clearly indicate which 4 problems you choose.** Each problem is worth 10 points You may do all the problems for up to five points extra credit.

I have chosen the following 4 problems: ______________________________

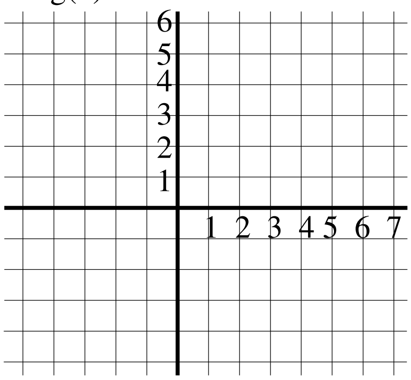
A.) Given the graph of y = g(x) below, draw the following graphs:



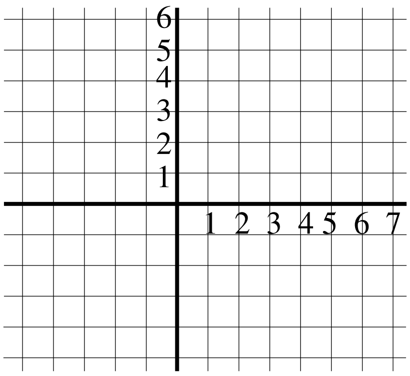
$y = g(x-1)$



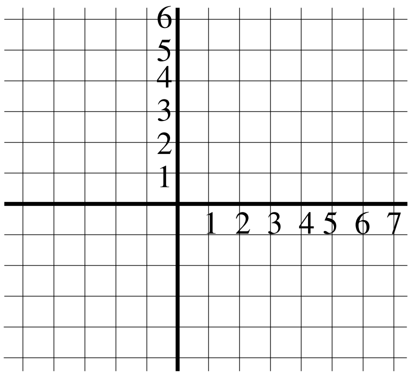
$y = \dfrac{1}{g(x)}$



$y = g^{-1}(x)$



$y = g'(x)$



B.) Use calculus to show that the equation $x^9 + 4x^3 + 10x = 0$ has at most one real root.

Let $f(x) = x^9 + 4x^3 + 10x$

Suppose $f(x)$ has two real roots. I.e, there exists a, b, $a \neq b$ such that $f(a) = 0 = f(b)$.

By the Mean Value Theorem (or Rolle's Thm), there exists c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$

Since $f(x) = x^9 + 4x^3 + 10x$, $f'(x) = 9x^8 + 12x^2 + 10$.

Thus $0 = f'(c) = 9c^8 + 12c^2 + 10$. But $9c^8 + 12c^2 + 10 > 0$.

Hence $f(x)$ has at most one real root.

[Note: could use the intermediate value thm (IVT) to show there is at least one real root, but the question didn't ask if there was a root. It only asked if there was more than one real root).

C.) Express the following integral as a limit of Riemann sums. Do not evaluate the limit: $\int_2^8 (x+1)sin(3x)dx$.

$\Delta x = \frac{8-2}{n} = \frac{6}{n}$ = width

$x_i = 2 + \frac{6i}{n}$. Thus $f(x_i) = (2 + \frac{6i}{n} + 1)sin(3(2 + \frac{6i}{n})) = (3 + \frac{6i}{n})sin(6 + \frac{18i}{n})$ = height

Answer C.) $\underline{lim_{n\to\infty} \Sigma_{i=1}^{n} [(3 + \frac{6i}{n})sin(6 + \frac{18i}{n})] \frac{6}{n}}$

D.) Water is leaking out of an inverted conical tank at a rate of 30 m^3/sec at the same time that water is being pumped into the tank at a constant rate. The tank has height 10 m and the diameter at the top is 6 m. If the water level is rising at a rate of 4 m/sec when the height of the water is 5m, find the rate at which water is pumped into the tank.

$V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt} = \frac{dV_{in}}{dt} - 30$

$\frac{dh}{dt} = 4$. When $h = 5$, $\frac{dV_{in}}{dt} =?$

$\frac{r}{h} = \frac{3}{10}$. Hence $r = \frac{3h}{10}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{3h}{10})^2 h = \frac{3\pi h^3}{100}$.

Thus $\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} = \frac{9\pi}{100}(5)^2(4) = 9\pi$

Answer D.) $\underline{(9\pi + 30)m^3/sec}$

E.) Use calculus to find the point on the line $y = 2x + 8$ that is closest to the point (4, 3). Explain why your answer is optimal.

minimize $d(x) = \sqrt{(x-4)^2 + (2x+8-3)^2} = \sqrt{(x-4)^2 + (2x+5)^2}$

$d'(x) = \frac{1}{2}[(x-4)^2 + (2x+5)^2]^{\frac{-1}{2}}[2(x-4)+2(2x+5)(2)] = \frac{2(x-4)+2(2x+5)(2)}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}} = \frac{2x-8+8x+20}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}}$

$= \frac{10x+12}{2[(x-4)^2+(2x+5)^2]^{\frac{1}{2}}} =$

Hence $d'(x)$ exists everywhere and if $d'(x) = 0$, then $x = -\frac{12}{10}$

If $x < -\frac{12}{10}$, then $d'(x) < 0$. If $x > -\frac{12}{10}$, then $d'(x) > 0$. Hence d is a decreasing function on $(-\infty, -\frac{12}{10})$ and is an increasing function on $(-\frac{12}{10}, \infty)$. Thus the minimum distance occurs when $x = -\frac{12}{10}$

When $x = -\frac{12}{10}$, $y = 2(\frac{-12}{10}) + 8 = \frac{-12}{5} + \frac{40}{5} = \frac{28}{5}$

Answer E.) $\underline{(-\frac{12}{10}, \frac{28}{5})}$

Final Exam Part C, Dec., 2005 SHOW ALL WORK
Math 25 Calculus I Either circle your answers or place on answer line.

Note the proof problems on this page are completely optional. You may choose to prove one (and only one) of the following statements. If you choose to do one of the following problems, it can replace your lowest point problem in the optional section OR 80% of your lowest point problem in the required section. If you chose to do one of the following problems, clearly indicate your choice.

I have chosen the following problem: ______________________________

I.) State and prove Rolle's theorem.

Rolles thm: If f is continuous on $[a,b]$ and f is differentiable on (a,b), and if $f(a) = f(b) = 0$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof: Suppose there exists an $x \in (a,b)$ such that $f(x) > 0$.

By the extreme value theorem, f attains its maximum value at some $c \in [a,b]$ Since there exists an $x \in (a,b)$ such that $f(x) > 0$, the maximum value does not occur at a or b. Hence f attains its maximum value at some $c \in (a,b)$ and thus f has a local maximum at c. Since f is differentiable on (a,b), $f'(c) = 0$.

Suppose there exists an $x \in (a,b)$ such that $f(x) < 0$.

By the extreme value theorem, f attains its minimum value at some $c \in [a,b]$ Since there exists an $x \in (a,b)$ such that $f(x) < 0$, the minimum value does not occur at a or b. Hence f attains its minimum value at some $c \in (a,b)$ and thus f has a local minimum at c. Since f is differentiable on (a,b), $f'(c) = 0$.

If $f(x) = 0$ for all $x \in [a,b]$, then $f'(x) = 0$ for all $x \in (a,b)$.

II.) State the ϵ, δ definition of limit and use it to prove that $lim_{x \to 3}(2x) = 6$.

$lim_{x \to a} f(x) = L$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$

Thus $lim_{x \to 3}(2x) = 6$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|2x - 6| < \epsilon$

Scratch work. Need $|2x - 6| < \epsilon$

$|2x - 6| = 2|x - 3|$

Proof: Take $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Suppose $0 < |x - 3| < \delta = \frac{\epsilon}{2}$. Then $2|x - 3| < \epsilon$. Hence $|2x - 6| < \epsilon$