4.7: Optimization

A right circular cylinder is inscribed in a sphere of radius 5m. Find the largest possible surface area of such a cylinder.

$S(x) = 2(\pi x^2) + (2\pi x)(2y) = 2(\pi x^2) + (4\pi x)\sqrt{25 - x^2}$

$S : [0, 5] \to \mathbf{R}$

Note we chose the domain to be a closed interval, so that we can apply the EVT. In reality, we can't have a cylinder of radius 0 ($x = 0$) or a cylinder of height 0 ($x = 5$). But we can find the maximum $S(x)$ value in the interval (0, 5) by finding the maximum $S(x)$ value in the interval [0, 5] **<u>if</u>** the maximum does not occur at $x = 0$ or $x = 5$.

$S'(x) = 4\pi x + 4\pi\sqrt{25 - x^2} + 4\pi x(\frac{1}{2})(25 - x^2)^{-\frac{1}{2}}(-2x) = 0$, DNE

DNE: $x = 5$

$0 = 4\pi x + 4\pi\sqrt{25 - x^2} - 4\pi x^2(25 - x^2)^{-\frac{1}{2}}$

$0 = x + \sqrt{25 - x^2} - x^2(25 - x^2)^{-\frac{1}{2}}$

$(\sqrt{25 - x^2})(0) = (\sqrt{25 - x^2})[x + \sqrt{25 - x^2} - x^2(25 - x^2)^{-\frac{1}{2}}]$

$0 = x\sqrt{25 - x^2} + 25 - x^2 - x^2$

$2x^2 - 25 = x\sqrt{25 - x^2}$

$(2x^2 - 25)^2 = (x\sqrt{25 - x^2})^2$. Note by squaring both sides, our new equation has more solutions than our previous equation. Hence not all solutions to this new equation will be critical points, but as long as we check all critical points, it's fine to check additional points if you are not sure which of the solutions to this new equation are critical points.

$4x^4 - 100x^2 + (25)^2 = x^2(25 - x^2)$

$5x^4 - 125x^2 + (25)^2 = 0.$ Hence $x^4 - 25x^2 + 125 = 0$

$x^2 = \frac{25 \pm \sqrt{(25)^2 - 4(125)}}{2} = \frac{25 \pm \sqrt{5(125) - 4(125)}}{2} = \frac{25 \pm \sqrt{125}}{2}$

$x = \pm\sqrt{\frac{25 \pm \sqrt{125}}{2}}.$ Note $\sqrt{\frac{25 \pm \sqrt{125}}{2}} \in [0, 5]$

$S : [0, 5] \to \mathbf{R}$. Thus we are not interested in negative solutions.

By the EVT, we need to check critical points and endpoints to see which point(s) gives the largest $S(x)$ value.

$S(x) = 2\pi x^2 + 4\pi x\sqrt{25 - x^2}$

Endpoints: $x = 0, 5$: $S(0) = 0,$ $S(5) = 50\pi$

Potential critical points in [0, 5]: $x = \sqrt{\frac{25 \pm \sqrt{125}}{2}}$

$S(\sqrt{\frac{25 \pm \sqrt{125}}{2}}) = 2\pi\left(\frac{25 \pm \sqrt{125}}{2}\right) + 4\pi\sqrt{\frac{25 \pm \sqrt{125}}{2}}\sqrt{25 - \frac{25 \pm \sqrt{125}}{2}}$

$= \pi\left(25 \pm \sqrt{125}\right) + 4\pi\sqrt{\frac{25 \pm \sqrt{125}}{2}}\sqrt{\frac{25 \mp \sqrt{125}}{2}}$

$= \pi\left(25 \pm \sqrt{125}\right) + 2\pi\sqrt{25^2 - 125} = \pi\left(25 \pm \sqrt{125}\right) + 2\pi\sqrt{25^2 - 5(25)}$
$= \pi\left(25 \pm \sqrt{125}\right) + 2\pi\sqrt{20(25)} = \pi\left(25 \pm 5\sqrt{5}\right) + 20\pi\sqrt{5}$

Hence the maximum surface area is $\pi\left(25 + 5\sqrt{5}\right) + 20\pi\sqrt{5} = 25\pi + 25\pi\sqrt{5}\, m^2$. This cylinder has radius $\sqrt{\frac{25+5\sqrt{5}}{2}}$m and height $= 2\sqrt{25 - x^2} = 2\sqrt{25 - \frac{25+\sqrt{125}}{2}} = 2\sqrt{\frac{25-5\sqrt{5}}{2}}$ m.

If we check, $S'(\sqrt{\frac{25-\sqrt{125}}{2}}) \neq 0$ and hence $\sqrt{\frac{25-\sqrt{125}}{2}}$ is not a critical point, but it is easier to check $S(\sqrt{\frac{25-\sqrt{125}}{2}})$ than to check if $\sqrt{\frac{25-\sqrt{125}}{2}}$ is a critical point.