$G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$, $E' \subset E$, and G' is a graph.

$G[V'] = (E', V')$ the subgraph of G induced or spanned by V' if $E' = \{xy \in E \mid x, y \in V'\}$.

$G' = (V', E')$ is a spanning subgraph of $G = (V, E)$ if $V' = V$.

$G - W = G[V - W]$, $G - E' = (V, E - E')$, $G + xy = (V, E \cup \{xy\})$ where x, y are nonadjacent vertices in V.

$|G|$ = order of $G = |V(G)|$ = number of vertices.

$e(G)$ = size of $G = |E(G)|$ = number of edges.

G^n is a graph of order n, $G(n, m)$ is a graph of order n and size m.

$E(U, V)$ = set of $U - V$ edges = set of all edges in $E(G)$ joining a vertex in U to a vertex in V where $U \cap V = \emptyset$.

The complement of $G = (V, E) = \overline{G} = (V, V^{(2)} - E)$

K_n = complete graph on n vertices. $E_n = \overline{K_n}$ = empty graph with n vertices. $K_1 = E_1$ is trivial.

$\Gamma(x) = \Gamma_G(x) = \{y \mid xy \in E(G)\}$.

$d(x) = d_G(x) = deg(x)$ = degree of $x = |\Gamma(x)|$.

$\delta(G) = min\{d(x) \mid x \in V(G)\}$.

$\Delta(G) = max\{d(x) \mid x \in V(G)\}$.

v is an isolated vertex if $d(v) = 0$.

$\Sigma_{x \in V} d(x) = 2e(G)$.

A walk in a graph, $W = v_0, e_1, v_1, e_2, ..., e_n, v_n$, where $v_i \in V$ and $e_i = v_{i-1}v_i \in E$.

length of $W = n$.

trail = walk with distinct edges.

circuit = closed trail.

path = walk with distinct vertices (= trail with distinct vertices).

cycle = circuit with distinct vertices.

A set of vertices (edges) is independent if no two elements in the set are adjacent

A set of paths is independent if no two paths share an interior vertex.

$d(x, y)$ = length of shortest $x - y$ path. If there is no $x - y$ path, then $d(x, y) = \infty$.

A graph is connected if given any pair of distinct vertices, x, y, there is an $x - y$ path.

A component of a graph = a maximal connected subgraph.

A cutvertex = a vertex whose deletion increases the number of components.

A bridge = an edge whose deletion increases the number of components.

A forest = an acyclic graph = a graph without any cycles.

A tree = a connected forest.

$G = (V, E)$ is bipartite if there exists vertex classes, V_1, V_2, such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, and $xy \in E$, $x \in V_i$ implies $y \notin V_i$ (i.e., no edge joins two vertices in the same class).

$K(n_1, ..., n_r)$ = complete r-partite graph. $K_{p,q} = K(p, q)$, $K_r(t) = K(t, t, ..., t)$

Thm 3: Suppose that $C = (W, E')$ is the component of $G = (V, E)$ containing the vertex x. Then $W = \{y \in V \mid G \text{ contains an } x - y \text{ path } \} = \{y \in V \mid G \text{ contains an } x - y \text{ trail } \}$ $= \{y \in V \mid d(x, y) < \infty\}$ = equivalence class of x where we take the smallest equivalence relation on V such that u is equivalent to v if $uv \in E$.

If $G = (V, E)$ connected, $W \subset V$, $G - W$ disconnected, then W *separates* G.

If $s, t \in V - W$ and s and t in different components of $G - W$, then W *separates s from t*.

Thm 5 (Menger 1927)
(i) Let s, t distinct nonadjacent vertices in G.
$min\{|W| \mid W \subset V(G),\ W \text{ separates } s \text{ from } t\}$ = maximum number of independent $s - t$ paths.

(ii)Let s, t distinct vertices in G.
$min\{|E'| \mid E' \subset E(G), E' \text{ separates } s \text{ from } t\}$ = maximum number of edge-disjoint $s - t$ paths.

Cor 6: For $k \geq 2$, G is k-connected iff $V(G) \geq 2$ and any two vertices can be joined by k independent paths. G is k-edge-connected iff $V(G) \geq 2$ and any two vertices can be joined by k edge disjoint paths.

If G_1, G_2 k-connected, $|V(G_1) \cap V(G_2)| \geq k$, then $G_1 \cup G_2$ is k-connected.

Pf: If $|W| \leq k - 1$, then $G_1 \cup G_2 - W = (G_1 - W) \cup (G_2 - W)$

A subgraph B of G is a *block of* G if B is a bridge or if B is a maximal 2-connected subgraph of G.

If B_1 and B_2 are blocks, then $|V(B_1) \cap V(B_2)| \leq 1$.

If $x, y \in V(B)$, a block, and $x \neq y$ then $G - E(B)$ does not contain an $x - y$ path.

A vertex v belongs to at least two blocks iff v is a cutvertex.

$E(G) = \cup_{i=1}^{p} E(B_i)$ where $E(B_i) \cap E(B_j) = \emptyset$ if $i \neq j$ and B_i's are blocks.

Suppose G nontrivial connected graph where $\{v_1, ..., v_n\}$ is the set of cutvertices of G. $bc(G)$ = block-cutvertex graph of $G = (V', E')$ where $V' = \{v_1, ..., v_n, B_1, ..., B_p\}$ and $E' = \{(v_i, B_j) \mid v_i \in B_j\}$.

$bc(G)$ is a tree.

An endvertex of $bc(G)$ is a block of G = *endblock* of G.

$G = (V, E)$ be a finite directed graph.

$\Gamma^+(x) = \{y \in V \mid xy \in E\}$

$\Gamma^-(x) = \{y \in V \mid yx \in E\}$

A *flow* f is a function $f : E \to [0, \infty)$ where $f(xy) = f(x, y)$.

A *flow* f from vertex s (the *source*) to a vertex t (the *sink*) is a function $f : E \to [0, \infty)$ where $f(xy) = f(x, y)$ such that if $x \in V - s, t$,

$$\Sigma_{y \in \Gamma^+(x)} f(x, y) = \Sigma_{z \in \Gamma^-(x)} f(z, x)$$

The *value of* f or the *amount of flow from* s *to* t = the net current leaving $s = \Sigma_{y \in \Gamma^+(s)} f(s, y) = \Sigma_{z \in \Gamma^-(t)} f(z, t)$= the net current flowing into t.

The capacity of an edge $c(x, y)$ is a non-negative number such that the current flowing through xy $= f(x, y) \leq c(x, y)$.

The set of directed $X - Y$ edges $= E(X, Y) = \{xy \in E \mid x \in X, y \in Y\}$.

If $g : E \to R$ is a function, $g(X, Y) = \Sigma_{xy \in E(X,Y)} g(x, y)$.

$E(S, E - S)$ is a *cut* separating s from t if $s \in S$ and $t \in E - S$.

The *capacity* of a cut $E(S, E - S) = c(S, E - S)$.
