Suppose D^* is the condensation of $D = (V, A)$. Let $K_1, ..., K_p$ be the strong components of D. Then $\{K_1, ..., K_p\}$ are the vertices of D^*.

Thm 2.7: D^* is acyclic.

Proof 1: Suppose D^* contains a cycle $K_{i_1}, ..., K_{i_m}$ where $K_{i_1} = K_{i_m}$. Then there exists $v_{j,b} \in K_{i_j}$, $v_{j+1,e} \in K_{i_{j+1}}$ such that $(v_{j,b}, v_{j+1,e}) \in A$. Since $v_{j,e}, v_{j,b}$ are in the same strong component, K_{i_j}, there exists a path $v_{j,e}, u_{j,1}, ..., u_{j,n_j}, v_{j,b}$. Similarly $v_{m,e}, v_{1,b}$ are in the same strong component, $K_{i_1} = K_{i,m}$, so there exists a path $v_{m,e}, u_{1,1}, ..., u_{1,n_1}, v_{1,b}$

Thus $v_{1,b}, v_{2,e}, u_{2,1}, ..., u_{2,n_2}, v_{2,b}, v_{3,e}, u_{3,1}, ..., u_{3,n_2}, v_{3,b}, ..., v_{m-1,b}, v_{m,e}, u_{1,1}, ..., u_{1,n_1}, v_{1,b}$ is a closed path in D. Hence there exists a path from $v_{1,b}$ to $v_{2,e}$ (namely $v_{1,b}, v_{2,e}$) and there is a path from $v_{2,e}, v_{1,b}$ (namely $v_{2,e}, u_{2,1}, ..., u_{2,n_2}, v_{2,b}, v_{3,e}, u_{3,1}, ..., u_{3,n_2}, v_{3,b}, ..., v_{m-1,b}, v_{m,e}, \mathrm{u}_{1,1}, ..., u_{1,n_1}, v_{1,b}$). Thus $v_{2,e} \in K_{i_1}$. But this is a contradiction by thm 2.6 since $v_{2,e} \in K_{i_2}$ and $K_{i_1} \neq K_{i_2}$ (since $K_{i_1}, ..., K_{i_m}$ is a cycle).

Note: using a lot of notation helped us to write the above proof. Often notation can help us in writing a proof and can also help our understanding. But proofs using less notation and more English are also valid and sometimes are easier to understand.

I will first simplify the above proof by using different notation. Instead of giving the path in terms of vertices, I can just give it a name, say $\mathbf{p_j}$. Thus in the following $\mathbf{p_j}$ will refer to the path $v_{j,e}, u_{j,1}, ..., u_{j,n_j}, v_{j,b}$ $\mathbf{p_1}$ will refer to the path $v_{m,e}, u_{1,1}, ..., u_{1,n_1}, v_{1,b}$.

Proof 2: Suppose D^* contains a cycle $K_{i_1}, ..., K_{i_m}$ where $K_{i_1} = K_{i_m}$. Then there exists $v_{j,b} \in K_{i_j}$, $v_{j+1,e} \in K_{i_{j+1}}$ such that $(v_{j,b}, v_{j+1,e}) \in A$. Since $v_{j,e}, v_{j,b}$ are in the same strong component, K_{i_j}, there exists a path $\mathbf{p_j}$ from $v_{j,e}$ to $v_{j,b}$. Similarly $v_{m,e}, v_{1,b}$ are in the same strong component, $K_{i_1} = K_{i,m}$, so there exists a path $\mathbf{p_1}$ from $v_{m,e}$ to $v_{1,b}$.

Thus $\mathbf{p_2 p_3}, ..., \mathbf{p_m p_1}$ is a closed path in D starting and ending at $v_{1,b}$. Since $v_{2,e}$ is a vertex in this closed path, there exists a path from $v_{1,b}$ to $v_{2,e}$ and there is a path from $v_{2,e}, v_{1,b}$

Thus $v_{2,e} \in K_{i_1}$. But this is a contradiction by thm 2.6 since $v_{2,e} \in K_{i_2}$ and $K_{i_1} \neq K_{i_2}$ (since $K_{i_1}, ..., K_{i_m}$ is a cycle).

Now with more English:

Proof 3: Suppose D^* contains a cycle $K_{i_1}, ..., K_{i_m}$ where $K_{i_1} = K_{i_m}$. Then there exists $v_{j,b} \in K_{i_j}$, $v_{j+1,e} \in K_{i_{j+1}}$ such that $(v_{j,b}, v_{j+1,e}) \in A$. Since $v_{j,e}, v_{j,b}$ are in the same strong component, K_{i_j}, there exists a path from $v_{j,e}$ to $v_{j,b}$. We can now form a closed path by joining these paths together. Since this path is closed, all the vertices in this path must be in the same strong component. Hence $v_{1,b} \in K_{1_j}$ and $v_{2,e} \in K_{i_2}$ are in the same strong component. Thus $K_{i_1} = K_{i_2}$, But this is a contradiction since $K_{i_1}, ..., K_{i_m}$ is a cycle.

Note the first proof has the most detail. If anyone questions your proof, you need to be able to provide more details. For example, in the 2nd and 3rd proof, I state that if two vertices are in the same closed path, then these vertices must be in the same strong component. This is true, but we don't have a theorem stating it. Thus it would be valid for someone to be unsure if it were true and to ask me to provide more detail. Sometimes additional notation is necessary to provide the detail. Sometimes more notation is not necessary.

NOTE: Sometimes things that seem to be obviously true are NOT true. Hence anything you state, you should be able to prove or refer to a theorem or definition. A general rule for what needs to be written in a proof is that is you can prove it, you don't need to (since you can prove it, you know it is true), but if you don't know how to prove it, then you must prove it (in case it is not true).

2.4: Digraphs and matrices

Recall if $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ matrices, then

$$A + B = (a_{ij} + b_{ij})$$
$$A \times B = (a_{ij} b_{ij})$$
$$AB = (c_{ij}) \text{ where } c_{ij} = \Sigma_{k=1}^{n} a_{ik} b_{kj} = row(i) \cdot column(j)$$

The following refer to the digraph $D = (V, A)$ where $V = (v_1, ..., v_n)$ where the vertices have been given a particular order.

Defn: The adjacency matrix of $D = (V, A)$ is

$$A(D) = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in A \\ 0 & \text{otherwise} \end{cases}.$$

Thm 2.11: If A is the adjacency matrix of D, then the i, jth entry of A^k is the number of paths of length k from v_i to v_j.

Defn: The reachability matrix of $D = R(D) = (r_{ij})$ where $r_{ij} = \begin{cases} 1 & \text{if } v_j \text{ is reachable from } v_i \\ 0 & \text{otherwise} \end{cases}.$

Defn: The Boolean function $b : \mathcal{M} \rightarrow \{0, 1\}$ where $\mathcal{M} = \{0, 1, 2, ...\}$ is defined by $b(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}.$ ∎

B : set of matrices with nonnegative integer entries $\rightarrow$ set of matrices with entries 0 and 1 is defined by $B(A) = (b(a_{ij}))$.

Thm 2.12: Suppose A is the adjacency matrix and R is the reachability matrix of the digraph D. If D has n vertices, then

$$R = B(I + A + A^2 + ... + A^{n-1}) = B((I + A)^{n-1})$$

$$I = \text{identity matrix} = (i_{kp}) \text{ where } i_{kp} = \begin{cases} 0 & \text{if } k \neq p \\ 1 & \text{if } k = p \end{cases}$$

$J = (j_{kp})$ where $j_{kp} = 1$ for all k, p.

$R' = (r_{ji})$ is the transpose of $R = (r_{ij})$.

Thm 2.13 (Ross and Harary 1959). Suppose A is the adjacency matrix and R is the reachability matrix of the digraph D. Let $n = \#$ of vertices of D. Then

a.) D is strongly connected iff $R = J$.

b.) D is unilaterally connected iff $B(R + R') = J$.

c.) D is weakly connected iff $B(I + A + A')^{n-1}) = J$.

Thm 2.14. $R =$ the reachability matrix of the digraph D. Let $R \times R' = (t_{ij})$. Let $R^2 = (s_{ij})$. Then

(a.) v_j is in the strong component containing the vertex v_i if and only if $t_{ij} = 1$.

(b.) # of vertices in the strong component $u_i = s_{ii}$.

Defn: The distance matrix of the graph G is the matrix (d_{ij}) where d_{ij} is the distance from vertex v_i to v_j.

Thm 2.15: d_{ij} is the smallest number k such that $B(A^k) = 1$.

HW#4 (due 2/20): 2.3: 1, 2, 3, 4, 6, 10, 11, 18 and give rough estimate of computational complexity of TSP.

HW#5 (due 2/27): 2.4: 1, 2, 3, 4, 5, 7, 8, 12 AND 13, 14, 15.