Math 541: Topics in Topology: Noncommutative Knot Theory: Knot Concordance

Fall 2009 TTh 10:50-12:05 HB 427 Professor Tim Cochran cochran@rice.edu

Prerequisites: Math 444, Math 445, Groups, Rings, modules, tensor products of modules and rings, Hom, Ext, and Tor.

Grade will be based on attendance, participation and occasional exercises. No textbook required.

1. Classical Knot Invariants

The set of knots (knotted circles) in S^3 under the equivalence relation of isotopy is an abelian monoid under connected sum. If one declares the set of *slice knots*, those that bound an embedded disk in B^4, to be zero then one obtains an abelian group, called the *classical knot concordance group*, $\mathcal{C}$, (There are actually two groups corresponding to two possible categories: SMOOTH and TOPOLOGICAL). Whereas knots are intimately related to the classification of 3-manifolds, the study of $\mathcal{C}$ is intimately related to the classification of 4-manifolds.

In this class we will discuss some of the recent advances in the study of $\mathcal{C}$. We will endeavor to present material disjoint from that in the textbook Rolfsen (which over half the class will have read), but there will be some overlap in the beginning.

We will begin by defining the Alexander module of a knot, its associated invariants and properties and the Blanchfield linking form of a knot and its properties. We will discuss *classical signatures* and, briefly, the *Arf invariant* of a knot.

Then we will discuss *knot concordance*, *ribbon knots*, and *slice knots*, which will entail require a short diversion into *handlebody theory* and a small bit of *Morse theory* (though I hope that a student will volunteer to talk about Morse theory in another seminar). Then we will discuss what information on concordance can be gleaned from the Alexander module and Blanchfield form. We will define a certain *Witt group* of forms, $\mathcal{AC}$, that is a quotient of $\mathcal{C}$, that encodes this information, and which is called the *algebraic knot concordance group.*

2. Higher-order Invariants and filtrations of $\mathcal{C}$

We will explain a strategy for filtering the group $\mathcal{C}$

$$\cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

which has the flavor of high-dimensional topology. It turns out that $\mathcal{C}/\mathcal{F}_{0.5} \cong \mathcal{AC}$, so that the classical invariants are truly "the tip of the iceberg", or more precisely, "the tip of a snowflake". We will define and discuss the *intersection form* on a 4-manifold and the *linking form on a* 3*-manifold.* We will define and discuss *homology, cohomology, Poincare Duality, intersection form and linking forms with arbitrary twisted coefficients.* We will discuss properties of signature: *additivity, signature defects.* We will also discuss von Neumann or $L^{(2)}$-signatures (this will be "blackboxed" until later in the course). At some point we make take a little break and discuss some "Kirby calculus", which is a tool for "drawing" and manipulating 4-manifolds. This is another topic that might profitably be discussed by a student in our VIGRE seminar.

Next we will discuss higher-order invariants of knots arising from linking forms and signature defects. This will involve some study of *modules over noncommutative rings* and *localization of modules over non-commutative rings.* This will also entail some homological algebra (some knowledge of which is assumed). We will prove that successive terms in this filtration are very large. *Series of groups*, especially the *derived series*, will play a large role.

At some point we will delve deeper into the definition of the von Neuman signature which will entail some discussion of some very basic functional analysis and the *von Neumann algebra associated to a discrete group.*