$f : A \to B$ is 1:1 iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

Hypothesis: $f(x_1) = f(x_2)$. Conclusion $x_1 = x_2$.

Hypothesis implies conclusion.
p implies q.
$p \Rightarrow q$.

Note a statement, $p \Rightarrow q$, is true if whenever the hypothesis p holds, then the conclusion q also holds.

To prove that a statement is true:
(1) Assume the hypothesis holds.
(2) Prove the conclusion holds.

Ex: To prove a function is 1:1:
(1) Assume $f(x_1) = f(x_2)$
(2) Do some algebra to prove $x_1 = x_2$.

$[p \Rightarrow q]$ is equivalent to $[\forall p, q \text{ holds}]$.

That is, for everything satisfying the hypothesis p, the conclusion q must hold.

A statement is false if the hypothesis holds, but the conclusion need not hold.

Hypothesis does not implies conclusion.
p does not imply q.
$$p \not\Rightarrow q.$$

That is there exists **a specific case** where the hypothesis holds, but the conclusion does not hold.

To prove that a statement is false:

Find an example where the hypothesis holds, but the conlusion does not hold.

Ex: To prove a function is not 1:1, find specific x_1, x_2 such that $f(x_1) = f(x_2)$, but $x_1 \neq x_2$.

Ex: $f : R \to R$, $f(x) = x^2$ is not 1:1
since $f(1) = 1^2 = 1 = (-1)^2 = f(-1)$, but $1 \neq -1$

$\sim [p \Rightarrow q]$ is equivalent to $\sim [\forall p,\ q$ holds$]$.

Thus if $p \Rightarrow q$ is false,
then it is not true that $[\forall p,\ q$ holds$]$.
That is, $\exists p$ such that q does not hold.

If $p \Rightarrow q$ is true, then
its contrapostive $\sim q \Rightarrow \sim p$ is also true.

But its converse, $q \Rightarrow p$ may not be true.

Thm 2.1.1: Pigeonhole Principle (weak form): If you have $n+1$ objects placed in n boxes, then at least one box will be occupied by 2 or more objects.

Thm 2.1.1: Pigeonhole Principle (weak form): If you have $n+1$ pigeons in n pigeonholes, then at least one pigeonhole will be occupied by 2 or more pigeons.

Thm 2.1.1: If $f : A \to B$ is a function and $|A| = n + 1$, and $|B| = n$, then f is not 1:1.

Cor: If $f : A \to B$ is a function and A is finite and $|A| > |B|$, then f is not 1:1.

Note that the domain must have more elements then the codomain to **guarantee** that f is not 1:1.

Recall the *converse* of [p implies q] is [q implies p].

Note the converse of a theorem is frequently false as the following example illustrates:

$$c : \{1, ..., n\} \to \{1, ..., n\},\ c(k) = 1 \text{ is not } 1:1,$$

but domain does not have more elements than the codomain.

$f : A \to B$ a function which is not 1:1 does not imply $|A| > |B|$.

Contrapositive of [p implies q] is [$\sim q$ implies $\sim p$]. The contrapositive of a theorem is true:

Cor: If $f : A \to B$ is a function which is 1:1, then $|A| \leq |B|$.

Related theorem:
Thm: If $f : A \to B$ is a function and
if $|A| = n = |B|$, then f is 1:1 iff f is onto.

Application 6: Chinese remainder theorem:
Suppose $m, n, a, b \in \mathcal{Z}$, $(m, n) = 1$, $0 \leq a \leq m-1$, $0 \leq b \leq n-1$, then $\exists x \geq 0$ such that $x = pm + a = qn + b$ for $p, q \in \mathcal{Z}$.

Moreover can take $p \in \{0, ..., n-1\}$.

Thm 2.2.1 Pigeonhole Principle (strong form): Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are put into n boxes, then for some i the ith box contains at least q_i objects

Proof Outline:

Cor: Pigeonhole Principle (weak form):

Proof. Let $q_i = 2$ for all i.

Cor: If $n(r-1)+1$ objects are put into n boxes, then there exists a box containing at least r objects.

Proof: Let $q_i = r$ for all i. Note $nr - n + 1 = n(r-1)+1$.

Cor A: If $m_i \in \mathcal{Z}_+$ and if $\frac{m_1+...+m_n}{n} > r-1$, then there exists an i such that $m_i \geq r$.

Cor A: If $m_i \in \mathcal{Z}_+$ and if $\frac{m_1+...+m_n}{n} \geq r$, then there exists an i such that $m_i \geq r$.

Lemma B: If $\frac{m_1+...+m_n}{n} < r$, then there exists an i s. t. $m_i < r$.

Appl: Suppose you have 20 pairs of shoes in your closet. If you grab n shoes at random, what should n be so that you are guaranteed to have a matching pair of shoes.

Appl: Suppose you have 20 pairs of socks. If you grab n socks at random, what should n be so that you are guaranteed to have a matching pair of shoes.

Appl: Suppose you have 20 pairs of socks. If 7 are black and 13 are white, and if you grab n socks at random, what should n be so that you are guaranteed to have a pair of socks of the same color.

Appl 7: If you have an arbitrary number of apples, bananas and oranges, what is the smallest number of these fruits that one needs to put in a basket in order to guarantee there are at least 8 apples or at least 6 bananas or at least 9 oranges in the basket.

Appl 9: Show that every sequence $a_1, a_2, ..., a_{n^2+1}$ contains either an increasing or decreasing subsequence of length $n+1$.

Example ($n = 2$):

$a_1 = 8$, $a_2 = 4$, $a_3 = 10$, $a_4 = 6$, $a_5 = 4$

Need $n + 1$ objects in our subsequence. Suppose $r = n + 1$.

Hence might need $n(r-1)+1 = n(n+1-1)+1 = n^2+1$ objects in n boxes in order to obtain at least $r = n+1$ objects in one of the boxes.

Let m_k = length of largest increasing subsequence beginning with a_k.

8	8, 10	$m_1 = 2$		
4	4, 10	4, 6	4, 4	$m_2 = 2$

10 $m_3 = 1$ **6** $m_4 = 1$ **4** $m_5 = 1$

Proof: Let m_k = length of largest increasing subsequence beginning with a_k, $k = 1, ..., n^2 + 1$.

Suppose there exists an $m_k \geq n+1$. Then there exists an increasing subsequence of length $m_k \geq n+1$. Hence there exists an increasing subsequence of length $n+1$.

Suppose $m_k < n+1$. Then $m_k = 1, 2, ...,$ or n.

Hence there exists an i such that $m_k = i$ for $n+1$ a_k's.

There exists $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ such that

$$m_{k_1} = m_{k_2} = ... = m_{k_{n+1}} = i$$

Show $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ is a decreasing sequence.

Suppose not. Then there exists a j such that $a_{k_j} > a_{k_{j+1}}$.

$\exists$ an increasing subsequence of length i starting at a_{k_j}

There does not exist an increasing subsequence of length $i+1$ starting at a_{k_j}

$\exists$ an increasing subsequence of length i starting at $a_{k_{j+1}}$

There does not exist an increasing subsequence of length $i + 1$ starting at $a_{k_{j+1}}$

Suppose $a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length i.

Then $a_{k_j}, a_{k_{j+1}}, a_{h_2}, a_{h_3}, ..., a_{h_i}$ is an increasing subsequence of length $i + 1$, a contradiction.