7.6: A geometry example

Thm 7.6.1: Let h_n = the number of ways of dividing a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals which do not intersect in the interior of the polygonal region. Define $h_1 = 1$. Then

$$h_n = \Sigma_{k=1}^{n-1} h_k h_{n-k}, \ n \geq 2$$

$$h_n = \frac{1}{n} \binom{2n-2}{n-1}, \ n \geq 1$$

$h_n = \Sigma_{k=1}^{n-1} h_k h_{n-k},\ n \geq 2,\ h_1 = 1,\ h_2 = h_1 h_1 = 1$

$h_3 = h_1 h_2 + h_2 h_1 = 2,\ h_4 = h_1 h_3 + h_2 h_2 + h_3 h_1 = 2 + 1 + 2 = 5$

Let $g(x) = 0 + h_1 x + h_2 x^2 + h_3 x^3 + ...$

Then $[g(x)]^2 = (h_1 x + h_2 x^2 + h_3 x^3 + ...)(h_1 x + h_2 x^2 + h_3 x^3 + ...)$

$= (h_1^2) x^2 + (h_1 h_2 + h_2 h_1) x^3 + (h_1 h_3 + h_2 h_2 + h_3 h_1) x^4 + ...$

$=$

7.7 Exponential Generating Functions

Instead of using $1, x, x^2, ..., x^n, ...$

we can use $\frac{1}{0!}, \frac{x}{1!}, \frac{x^2}{2!}, ..., \frac{x^n}{n!}, ...$

Ex: The standard generating function of 1, 1, 1, ... is

$g(x) = 1 + x + x^2 + ... + x^n + ... = \frac{1}{1-x}$

The exponential generating function of 1, 1, 1 is

$g^{(e)}(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}, ... = e^x$

Ex: The standard generating function of 2, 3, 4, 5, 0, 0, ... is

$g(x) = 2 + 3x + 4x^2 + 5x^3$

The exponential generating function of 2, 3, 4, 5, 0, 0, ... is

$g^{(e)}(x) = 2 + 3x + 4\frac{x^2}{2!} + 5\frac{x^3}{3!}, ...$

Ex: The standard generating function of $1, a, a^2, ...$ is

$g(x) = 1 + ax + a^2x^2 + ... + a^nx^n + ... = \frac{1}{1-ax}$

The exponential generating function of $1, a, a^2, ...$ is

$g^{(e)}(x) = 1 + ax + a^2\frac{x^2}{2!} + ... + a^n\frac{x^n}{n!}, ... = e^{ax}$

Thm 7.7.1: Let the multiset $S = \{n_1 \cdot a_1, n_2 \cdot a_2, ..., n_k \cdot a_k\}$ Let $h_n =$ the number of n-permutations of S.

$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) ... f_{n_k}(x)$ where

$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^{n_i}}{n_i!}$

Suppose a code consisting of integers between 0 and 5 inclusive of size k must contain the following:

even number of 0's

odd number of 1's

three or four 2's

the number of 3's is a multiple of five

between zero to four (inclusive) 4's

zero or one 5

Find the number of codes of size k.

Find the number of codes of size 100.