6.3 Derangements

Suppose each person in a group of n friends brings a gift to a party. In how many ways can the n gifts be distributed so that each person receives one gift and no person receives their own gift.

Let the set of friends $= \{p_1, ..., p_n\}$ where $p_j =$ person j.
Let the set of gifts $= \{g_1, ..., g_n\}$ where $g_j =$ the gift brought by person j.

Suppose $f : \{p_1, ..., p_n\} \to \{g_1, ..., g_n\}$,
$f(p_k) = g_j$ iff person p_k receives give g_j, the gift brought by person j.
If each person receives one gift, then f is a bijection.
If no person receives their own gift. Then $f(p_j) \neq g_j$.

In simpler notation,
$f : \{1, ..., n\} \to \{1, ..., n\}$ such that $f(j) \neq j$

Recall:
a permutation on $\{1, ..., n\}$ is a bijection $f : \{1, ..., n\} \to \{1, ..., n\}$ $\blacksquare$

Ex: The permutation 1 2 3 4 5 corresponds to the identity function.

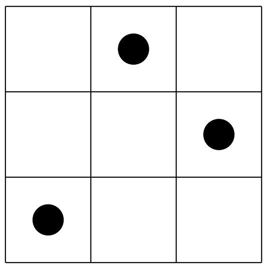
Ex: The permutation 1 3 2 corresponds to the function
$f(1) = 1,\ f(2) = 2,\ f(3) = 2$

Defn: A *derangement* of $\{1, ..., n\}$ is a permutation $i_1 i_2 ... i_n$ such that $i_j \neq j$. I.e, j is not in the jth place.

In function notation: $f(j) = i_j$, then if $i_1 i_2 ... i_n$ is a derangement, $f(j) \neq j$.

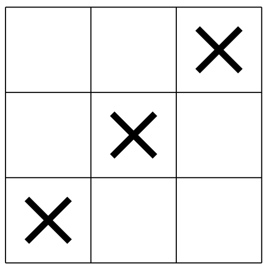
In yet other wording, recall a permutation corresponds to the placement of n non-attacking rooks on an $n \times n$ chessboard.

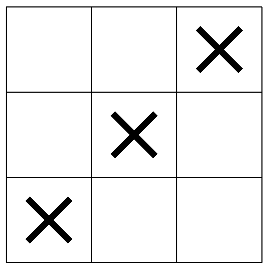
Ex: The permutation 1 3 2 corresponds to the following rook placement:



A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$. $\blacksquare$

Ex: If rooks are placed on the following 3×3 chessboard in non-attacking position, then the rook placement corresponds to a derangement if no rook is placed in a spot marked with an X.





Thus the derangements of $\{1, 2, 3\}$ are 2 3 1 and 3 1 2.

Let D_n = the number of derangements of $\{1, ..., n\}$.
Thus $D_3 = 2$.

Thm 6.3.1: For $n \geq 1$, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!})$

Pf: Use the inclusion and exclusion principle: If $A_i \subset S$,
$\overline{\cup A_i} = |S| - \Sigma_{j=1}^n |A_j| + \Sigma_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$.

Choose S. What can we count which contains the set of derangements?

Let S = the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Choose A_j such that the set of derangements $= \overline{\cup A_j}$.
Let A_j = set of permutations such that j is in the jth spot.

$|A_j| = (n-1)!$ since there is only one choice for the jth spot (namely j), leaving $n-1$ terms to permute in the remaining $n-1$ places.

$|A_i \cap A_j| = (n-2)!$ since there is only one choice for the ith spot (namely i) and only one choice for the jth spot (namely j), leaving $n-2$ terms to permute in the remaining $n-2$ places.

Similarly, $|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = (n-k)!$.

Thus $D_n = n! - \Sigma_{j=1}^n (n-1)! + \Sigma_{i,j} (n-2)! - ... + (-1)^n (n-n)!$

$$= \binom{n}{0} n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - ... + \binom{n}{n} (-1)^n 0!$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} + ... + (-1)^n \frac{n!}{n!} = n!(1 - \frac{1}{1!} + \frac{1}{2!} + ... + (-1)^n \frac{1}{n!})$$

Recall $\binom{n}{k}$ = number of ways to choose k A_i's.

Sidenote: Finding the number of derangements is often called the hat check problem, because in the old days it was sometimes stated in the following terms: If n men check their hats, what is the probability that the hats are returned so that no one received their own hat.

Recall: If $E \subset S$, then the probability of $E = P(E) = \frac{|E|}{|S|}$

$S =$ sample space, $E =$ events.

Note: we assume each outcome is equally likely.

Suppose 4 customers at a restaurant order 4 meals. What is the probability that a waiter delivers these 4 orders to the 4 customers so that no customer receives what they ordered?

Answer: $\frac{D_4}{4!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24}$

The probability that a permutation of $\{1, ..., n\}$ is a derangement $= \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} + ... + (-1)^n \frac{1}{n!}$

Recall Taylor's expansion from Calculus I,
$f(x) = \Sigma_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x - a)^j$ (under appropriate hypothesis).

Thus $e^{-1} = \Sigma_{j=0}^{\infty}(-1)^j \frac{1}{j!}$ (let $f(x) = e^x, x = -1, a = 0$).

Thus e^{-1} is a good approximation for the probability of a derangement for n (slightly) large.

Thus the probability of a derangement is about the same when $n = 5$ as it is for $n = 50000000000$.

We can derive a recursive formula for D_n (we will look at many recursive formulas in chapter 7).

Lemma A: $D_n = (n-1)(D_{n-2} + D_{n-1})$ for $n \geq 3$.

Note the above formula is a recursive formula as we can determine D_n by calculating D_k for $k < n$.

Note $D_1 = 0$, $D_2 = 1$ (as 2 1 is the only derangement of $\{1, 2\}$).

Thus $D_3 = 2(0+1) = 2$, $D_4 = 3(1+2) = 9$, $D_5 = 4(2+9) = 44$, etc.

Combinatorial proof of lemma A:

Let $\mathcal{D}_n =$ the set of derangements of $\{1, ..., n\}$.

$D_n =$ the number of derangements of $\{1, ..., n\} = |\mathcal{D}_n|$.

We need to show that D_n is a product of $n-1$ and $D_{n-2}+D_{n-1}$. If we can partition $\mathcal{D}_n$ into $n-1$ subsets where each subset has $D_{n-2} + D_{n-1}$ elements, we can use the multiplication principle to show $D_n = (n-1)(D_{n-2} + D_{n-1})$.

Let's focus on one of the positions of a derangement. The last (nth) position of our derangement can be anything except n. Thus there are $n-1$ choices for the last (nth) position. Note the factor $n-1$ appears in our formula.

Let $\mathcal{R}_k =$ the set of derangements of $\{1, ..., n\}$ where k is in the nth position for $k = 1, ..., n-1$.

Then $\mathcal{D}_n = \cup_{j=0}^{n-1} \mathcal{R}_n$

Let $r_k = |\mathcal{R}_k|$ the number of derangements such that k is in the nth position.

Note that $r_1 = r_2 = ... = r_{n-1}$ (while $r_n = 0$).

Then $D_n = r_1 + ... + r_{n-1} = r_{n-1} + ... + r_{n-1} = (n-1)r_{n-1}$.

Thus we have (hopefully) simplified our problem to showing that $D_{n-2} + D_{n-1} = r_{n-1} =$ the number of derangements such that $n-1$ is in the nth position.

We need to partition the permutations in $\mathcal{R}_{n-1}$ into two sets, one with D_{n-2} elements and the other with D_{n-1} elements.

We can easily take care of D_{n-2}. The numbers $n-1$ and n do not appear in any derangement of $\{1, ..., n-2\}$. In $\mathcal{R}_{n-1}$, $n-1$ appears in the last position. We can take a look at the derangements in $\mathcal{R}_{n-1}$, such that n appears in the $(n-1)$st position. If we remove the nth and $(n-1)$st entries, we obtain a derangement in $\mathcal{D}_{n-2}$.

Ex: for $n = 5$, $23154 \in \mathcal{R}_{n-1} \to 231 \in \mathcal{D}_{n-2}$.

Thus $D_{n-2} =$ the number of derangements of $\mathcal{R}_{n-1}$ (such that $n-1$ is in the nth position and) n is in the $(n-1)$st position.

We can now look at the remaining derangements in $\mathcal{R}_{n-1}$ where n is not in the $(n-1)$st position.

Let $\mathcal{P}_n$ the set of derangement where $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \neq n, n-1$ (I.e, $k \leq n-2$).

We would like to show that $D_{n-1} = |\mathcal{P}_n|$ = the number of derangements of $\{1, ..., n\}$ such that $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \leq n-2$

Let $\mathcal{D}_{n-1}$ = the set of derangements of $\{1, ..., n-1\}$.

We would like to create a bijection from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$

Note that the differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. A derangement in $\mathcal{P}_n$ has n terms, while a derangement in $\mathcal{D}_{n-1}$ has $n-1$ terms. Thus we need to remove a term to go from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$.

If $i_1 i_2 ... i_n \in \mathcal{P}_n$, then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$. Also $i_j = n$ for some j.

In $\mathcal{D}_{n-1}$, $i_{n-1} = k$ for some $k \leq n-2$ (by definition of derangement of $\{1, ..., n-1\}$, so we have no problems with the $(n-1)$st term.

However, we have the following differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$:
$i_1 i_2 ... i_n$ has n terms and
n appears somewhere in $i_1 i_2 ... i_n$, and
$i_n = n-1$, so the placement of $n-1$ doesn't vary.
We can fix this by removing the nth term and replacing $i_j = n$ with $i_j = n-1$

Let $i_1 i_2 ... i_n \in \mathcal{P}_n$. Then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$.

Create $a_1 a_2 ... a_{n-1}$, a derangement of $\{1, ..., n-1\}$ by let $a_l = \begin{cases} i_l & \text{if } i_l \neq n,\ 1 \leq l \leq n-1 \\ n-1 & \text{if } i_l = n \end{cases}$

Ex: For $n = 5$, $25314 \in \mathcal{P}_n \to 2431 \in \mathcal{D}_{n-1}$.

This gives us a bijection between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. Thus $D_{n-1} = |\mathcal{P}_n|$.

Thus we have shown that $D_n = (n-1)r_{n-1} = (n-1)(D_{n-2} + |\mathcal{P}_n|) = (n-1)(D_{n-2} + D_{n-1})$ for $n \geq 3$.

Another (simpler) recurrance relation:

Lemma B: $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$

Proof by induction on n.

$n = 2$: $D_2 = 1$ (use definition or Thm 6.3.1)
$2D_1 + (-1)^2 = 2(0) + 1 = 1$.
Thus $D_n = nD_{n-1} + (-1)^n$ holds for $n = 2$.

Suppose $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.

By lemma A, $D_k = (k-1)D_{k-2} + (k-1)D_{k-1}$

By the induction hypothesis, $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.
Thus $(k-1)D_{k-2} = D_{k-1} - (-1)^{k-1} = D_{k-1} + (-1)^k$

Thus $D_k = D_{k-1} + (-1)^k + (k-1)D_{k-1} = kD_{k-1} + (-1)^k$

6.4 Permutations with Forbidden Positions

Goal: To **derive** a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$. In this section, we will cover arbitrary forbidden positions.

Let $X_j \subset \{1, ..., n\}$ for $j = 1, ..., n$.

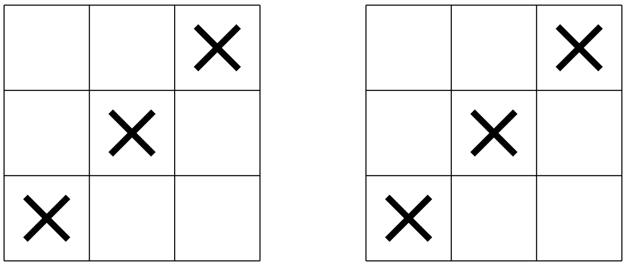
Defn: $P(X_1, X_2, ..., X_n)$ = the set of permutations $i_1 i_2 ... i_n$ of $\{1, ..., n\}$ such that $i_j \notin X_j$.

Defn: $p(X_1, X_2, ..., X_n) = |P(X_1, X_2, ..., X_n)|$

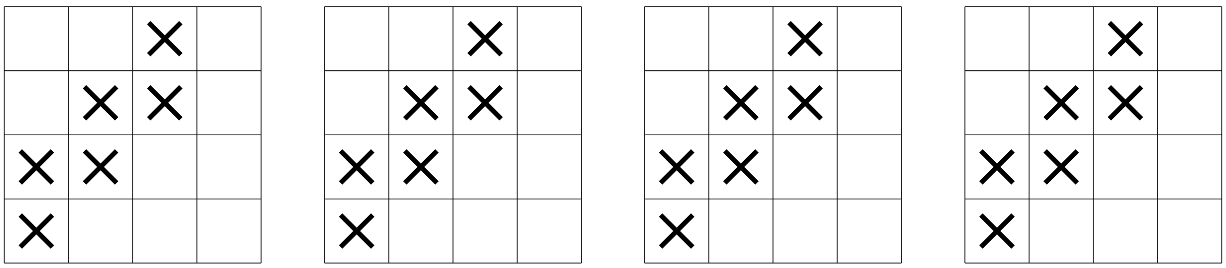
Ex: $P(X_1, X_2, ..., X_n)$ corresponds to the set of derangements of $\{1, ..., n\}$ if $X_j = \{j\}$. Thus $D_n = |P(\{1\}, \{2\}, ..., \{n\}|$

Recall, we can visualize permutations with forbidden positions via $n \times n$ chessboards.

Ex: Derangements of $\{1, 2, 3\}$:
$X_j = \{j\}$.



Non-derangement example:
$n = 4$, $X_i = \{j, j + 1\}$, $j = 1, 2, 3$, $X_4 = \emptyset$.



$$P(X_1, X_2, ..., X_n) = P(\{1, 2\}, \{2, 3\}, \{3, 4\}, \emptyset)$$
$$= \{3124, 3412, 3421, 4123\}.$$

$$p(X_1, X_2, ..., X_n) = p(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= |\{3124, 3412, 3421, 4123\}| = 4.$$

We can use the inclusion-exclusion principle to calculate $p(X_1, X_2, ..., X_n)$ (although in many cases, the computation can be tediously long and beyond computer capabilities for large n).

Similar to the proof of Thm 6.3.1. By the inclusion-exclusion principle,

$$p(X_1, X_2, ..., X_n) =$$
$$|S| - \Sigma_{j=1}^n |A_j| + \Sigma_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$$

where

Let $S =$ the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let $A_j =$ set of permutations $i_1 i_2 ... i_n$ such that $i_j \in X_j$ (for a fixed j).

Note there are $|X_j|$ ways to place a rook in the jth position. There are $(n-1)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to A_j.

Thus $|A_j| = |X_j|(n-1)!$.
$\Sigma_{j=1}^n |A_j| = \Sigma_{j=1}^n |X_j|(n-1)! = (n-1)! \Sigma_{j=1}^n |X_j| = r_1(n-1)!$
where $r_1 = \Sigma_{j=1}^n |X_j|$.

Note $r_1 =$ number of ways to place 1 nonattacking rooks on an $n \times n$ chessboard so that the rook is in a forbidden position.

Let's now look at $A_j \cap A_k$. $i_1 i_2 ... i_n \in A_j \cap A_k$, then $i_j \in X_j$ and $i_k \in X_k$. Thus there are $|X_j|$ ways to place a rook in the

jth position and $|X_k|$ ways to place a rook in the kth position. There are $(n-2)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to $A_j \cap A_k$.

Thus $|A_j \cap A_k| = |X_j||X_k|(n-2)!$.
$\Sigma_{i,j}|A_i \cap A_j| = \Sigma_{i,j}|X_j||X_k|(n-2)! = (n-2)!\Sigma_{i,j}|X_j||X_k|$. Let $r_2 = \Sigma_{i,j}|X_j||X_k|$.

Note $r_2 =$ number of ways to place 2 nonattacking rooks on an $n \times n$ chessboard so that each of the 2 rooks is in a forbidden position.

Similarly, define $r_k =$ number of ways to place k nonattacking rooks on an $n \times n$ chessboard so that each of the k rooks is in a forbidden position.

Then $\Sigma|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = r_k(k-1)!$.

Thus we have proved:

Thm 6.4.1: $p(X_1, X_2, ..., X_n) = n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n$.

Note that if there are many forbidden positions, then r_k may be difficult to calculate and it may be easier to calculate $p(X_1, X_2, ..., X_n)$ directly. If there are few forbidden positions, Thm 6.4.1 is the easier method to compute $p(X_1, X_2, ..., X_n)$.

Example: Let $X = \{1, 2, 3\}$. Let $i_1 i_2 i_3 \in P(\{1, 2\}, \{1, 3\}, \{3\})$. Then there is only one choice for both i_1 and i_2, namely $i_1 = 3$ and $i_2 = 2$. But this leaves only one choice for $i_3 = 1$. Thus

	×	×
×		
×	×	

$p(\{1,2\},\{1,3\},\{3\}) = 1.$

Note in this case, it was easiest to count directly and not use Thm 6.4.1.

Examples:

Let $X = \{1,2,3,4,5,6,7\}$. Calculate $P(\{1,2\},\{1,3\},\{3\},\{5\},\{4\},\emptyset,$

	×	×				
×						
×	×					
				×		
			×			

$r_1 =$ the number of forbidden positions $= 7$.

Note that the forbidden positions can be partitioned into two independent sets. Let $F_1 =$ the forbidden positions in the upper left 3×3 corner. Let F_2 contain the other two forbidden positions in the 2×2 square. These positions are independent because a rook in F_1 cannot attack a rook in F_2 (and vice versa).

To calculate r_2 we break the problem into the following cases:

Case 1: both rooks are in F_1

Subcase 1: one rook is placed in the 3rd column. There is only 1 possible placement for a rook in the 3rd column (1st row). There are 3 possible placements for the second rook so that the two rooks are in F_1 in non-attacking position.

Subcase 2: both rooks are place in the first two columns. In this case, the placement possibilities correspond to 13, 21, 23. 3

3 +3 = 6

Case 2: both rooks could be in F_2

Since there are only 2 rooks in F_2, there is only one way to place both rooks in F_2.

Case 3: one rook is in F_1 while the other rook is in F_2.

There are 5 + 2 = 7 ways to place one rook in F_1 and one rook in F_2.

Thus $r_2 = 6 + 1 + 7$

To calculate r_3 we break the problem into the following cases:

Case 1: all 3 rooks are in F_1:

number of

Case 2: 2 rooks in F_1, 1 rook in F_2.

Case 3: 1 rook in F_1, 2 rooks in F_2.

To calculate r_4 we break the problem into the following cases:

Case 1: 3 rooks are in F_1, 1 rook in F_2:

number of

Case 2: 2 rooks in F_1, 2 rook in F_2.

Case 3: 1 rook in F_1, 2 rooks in F_2.

To calculate r_5 we note that if we have 5 nonattacking rooks in forbidden positions, 3 rooks are in F_1 and 2 rook in F_2:

6.5 Another Forbidden Position Problem

Goal: To **derive** a formula for counting the number of permutations with relative forbidden positions.

Ex: Suppose children 1, 2, 3, 4, and 5 sit in a row in class. Children 1 and 2 cannot sit next to each other or they will cause trouble.

The order in which the children sit corresponds to a permutation of $\{1, 2, 3, 4, 5\}$. If 1 is in the ith spot, then 2 cannot be in the $i-1$st spot or the $i+1$th spot. Thus the pattern 21 or 12 cannot appear in our permutation. This is called a relative forbidden position as certain positions for the placement of 2 are forbidden, but these forbidden positions depend on the placement of 1.

We will focus on the relative forbidden position problem in which

Let Q_n = the number of permutations of $\{1, 2, ..., n\}$ in which none of the patterns 12, 23, 34, ..., $(n-1)n$ occurs.

Thm 6.5.1 $Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! -$

$... + \binom{n-1}{n-1}(-1)^{n-1}1!$

Proof: Use inclusion-exclusion principle.

Let $S =$ the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let $A_j =$ set of permutations which contain the pattern $j(j+1)$ for $j = 1, ..., n-1$.

To determine $|A_j|$, we can first look at all the permutations of $\{1, 2, ..., j, j+2, ..., n\}$. Since this set has $n-1$ elements, the number of permutations of $\{1, 2, ..., j, j+2, ..., n\} = n!$. Note that these permutations are in 1-1 correspondence with A_j as any permutation in A_j can be formed from a permutation of $\{1, 2, ..., j, j+2, ..., n\}$ by inserting $j+1$ right after j. Thus $|A_j| = (n-1)!$

Claim: $|A_i \cap A_j| = (n-2)!$ if $1 \leq i < j \leq n$

Suppose $|i-j| \geq 2$. In this case, $A_i \cap A_j$ is in 1-1 correspondence with permutations of a set of $n-2$ elements as we can create any permutation of $\{1, 2, ..., i, i+2, ..., j, j+2, ..., n\}$ by taking a permutation in $A_i \cap A_j$ and deleting $i+1$ and $j+1$. Thus $|A_j| = (n-2)! in this case.$

Suppose $|i-j| = 1$. Thus $j = i+1$ (since we assumed $i < j$). In this case, $A_i \cap A_j$ is the set of permutations which contain the pattern $ii+1$ and the pattern $i+1i+2$. Thus $A_i \cap A_j$ is the set of permutations which contain the pattern $ii+1i+2$. Thus $A_i \cap A_j$ is in 1-1 correspondence with permutations of a set of $n-2$ elements as we can create any permutation of $\{1, 2, ..., i, i+3, ..., n\}$ by taking a permutation in $A_i \cap A_j$ and deleting $i+1$ and $j+2$. Thus $|A_j| = (n-2)!$ in this case as well.

Similarly, $|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = (n-k)!$.

Thus by inclusion-exclusion $Q_n = n! - \Sigma_{j=1}^{n-1}(n-1)! + \Sigma_{i,j}(n-2)! - ... + (-1)^n(n-n)!$

$= \begin{pmatrix} n-1 \\ 0 \end{pmatrix} n! - \begin{pmatrix} n-1 \\ 1 \end{pmatrix} (n-1)! + \begin{pmatrix} n-1 \\ 2 \end{pmatrix} (n-2)! - ... + \begin{pmatrix} n-1 \\ n-1 \end{pmatrix} (-1)^{n-1} 1!$

$= n! - \frac{n!}{1!} + \frac{n!}{2!} + ... + (-1)^n \frac{n!}{n!} = n!(1 - \frac{1}{1!} + \frac{1}{2!} + ... + (-1)^n \frac{1}{n!})$

Recall $\begin{pmatrix} n-1 \\ k \end{pmatrix}$ = number of ways to choose k A_i's where $1 \leq i \leq n-1$.