decreasing $a_i \leq a_j$ for $j > i$

Appl 9: Show that every sequence $a_1, a_2, ..., a_{n^2+1}$ contains either an increasing or decreasing subsequence of length $n+1$.

8, 8, 4, 7, 4

Ex $(n = 2)$: $a_1 = 8$, $a_2 = 8$, $a_3 = 4$, $a_4 = 7$, $a_5 = 4$

Let m_k = length of largest increasing subsequence beginning with a_k.

$a_1 =$	**8**	8, 8		$m_1 = 2$
$a_2 =$	**8**	8		$m_2 = 1$
$a_3 =$	4	4, 7	4, 4	$m_3 = 2$
$a_4 =$	**7**	7		$m_{\cancel{2}\,4} = 1$
$a_5 =$	**4**	4		$m_{\cancel{2}\,5} = 1$

Need $n+1$ objects in our subsequence.
Suppose $r = n+1$.

Hence might need $n(r-1)+1 = n(n+1-1)+1 = n^2+1$ objects in n boxes in order to obtain at least $r = n+1$ objects in one of the boxes.

Want subseq of length $n+1 = 2+1 = 3$ for $n=2$
inc/dec

Proof: Let m_k = length of largest increasing subsequence beginning with a_k, $k = 1, ..., n^2+1$.

n^2+1 objects

Objects: The multiset $\{m_k \mid k = 1, ..., n^2+1\}$

Boxes: The value of m_k. ~ Right now n^2+1 boxes

Case 1: Suppose there exists an $m_k \geq n+1$.

Then there exists an increasing subsequence of length $m_k \geq n+1$. Hence there exists an increasing subsequence of length $n+1$.
starts at a_k

Case 2: Suppose $m_k < n+1$. Then $m_k = 1, 2, ...,$ or n. Thus we have n boxes.
box

Hence there exists an i such that $m_k = i$ for $n+1$ a_k's.
n

There exists $a_{k_1}, a_{k_2}, ..., a_{k_{n+1}}$ such that
$$m_{k_1} = m_{k_2} = ... = m_{k_{n+1}} = i$$

not decreasing

Claim: $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ is a decreasing sequence.

Suppose not.
Then there exists a j such that $a_{k_j} < a_{k_{j+1}}$.

~~$\exists$ an increasing subsequence of length i starting at a_{k_j}~~

There does not exist an increasing subsequence of length $i+1$ starting at a_{k_j}

$\exists$ an increasing subsequence of length i starting at $a_{k_{j+1}}$

~~There does not exist an increasing subsequence of length $i+1$ starting at $a_{k_{j+1}}$~~

Suppose $a_{k_{j+1}}, a_{h_2}, a_{h_3}, \ldots, a_{h_i}$ is an increasing subsequence of length i.

Then $a_{k_j}, a_{k_{j+1}}, a_{h_2}, a_{h_3}, \ldots, a_{h_i}$ is an increasing subsequence of length $i+1$, a contradiction.

$\Rightarrow$ Assumption is false

$\Rightarrow$ $a_{k_1}, \ldots a_{k_{n+1}}$ is a decrea subseq

Let K_n = the complete graph on n vertices.

That is $K_n = (V, E)$, where
V = the vertices of $K_n = \{v_1, \ldots, v_n\}$,
E = the edges of $K_n = \{\{v_i, v_j\} \mid 1 \leq i < j \leq n\}$

$K_1 =$ $K_2 =$ $K_3 =$  $K_4 =$ 

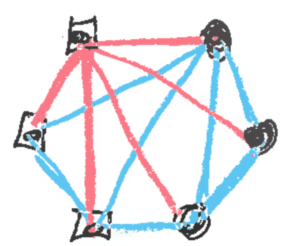
=

$K_5 =$  $K_6 =$ see next page

Example of a Ramsey theorem: In a group of 6 people, there are either 3 who know each other or 3 who are strangers to each other. $= r(3,3)$

Ramsey number $= r(s,t) = \underline{min}\{n \mid$ if the edges of K_n are colored red and blue, then there exists either a red K_s or a blue $K_t\}$

$r(3,3) = 6 \quad r(s,t) = r(t,s) \quad r(s,2) = r(2,s) = s$



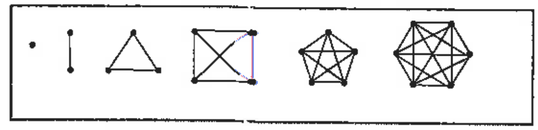
http://mail.baylorschool.org/~dkennedy/Numb3rs.ppt

***NUMB3RS* Activity: A Party of Six**

Episode: "Protest"

Topic: Graph Theory and Ramsey Numbers
Grade Level: 8 - 12
Objective: To see how a complete graph with edges of two colors can be used to model acquaintances and non-acquaintances at a party.
Time: About 30 minutes
Materials: Red and blue pencils or markers, paper

The first six complete graphs:



People = vertices

If two people (A and B) are at a party, there are only two possibilities: either A and B know each other, or A and B do not know each other. Draw the two possible graphs below.

color edge blue if A knows B & B knows A

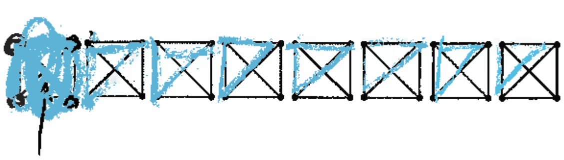
color edge red if A & B don't know each other

Draw all of the possible 3-person party graphs for A, B, and C below.

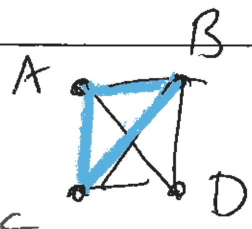
All possible colorings of K_3 using red & blue

A doesn't know B OR B " " A

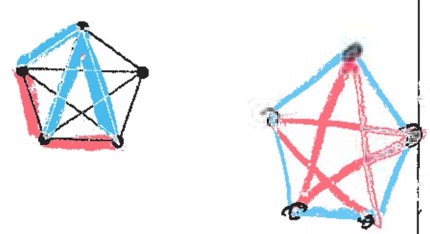
There are 64 possible 4-person party graphs for guests A, B, C, and D (Why?), but you will not be asked to draw them all. Instead, draw the 8 possible 4-person party graphs in which A, B, and C all know each other. We say A, B, and C are *mutual acquaintances.*



not a vertex



It is actually possible to color the edges of a 5-person party graph in such a way that there are neither three people that are mutual acquaintances nor three people that are mutual non-acquaintances. Can you do it?



$r(3,3) > 5$

red K_3

blue K_3

It is an interesting fact that every party of 6 people *must* contain either three mutual acquaintances or three mutual non-acquaintances.

Start with guest A.

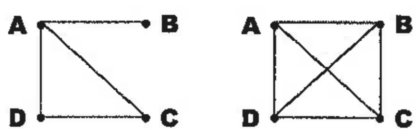
Among the remaining 5 guests, A has either at least three acquaintances or at least three non-acquaintances.

Case 1: Suppose A has three acquaintances: B, C, D.

If any two of these are acquainted, we have three mutual acquaintances.

If no two of these are acquainted, we have three mutual non-acquaintances!

Case 2: Suppose A has three non-acquaintances: B, C, D.



If any two of these are non-acquainted, we have three mutual non-acquaintances.

If no two of these are non-acquainted, we have three mutual acquaintances!

The *Ramsey Number R*(m, n) gives the minimum number of people at a party that will guarantee the existence of either m mutual acquaintances or n mutual non-acquaintances.

We just constructed a proof that $R(3, 3) = 6$.

Ramsey's Theorem guarantees that $R(m, n)$ exists for any m and n.

Intriguingly, there is still no known procedure for *finding* Ramsey numbers!

It has actually been known since 1955 that $R(4, 4) = 18$.

We do not know $R(5, 5)$, but we do know that it lies somewhere between 43 and 49.

All we really know about $R(6, 6)$ is that it lies somewhere between 102 and 165.

There is a cash prize for finding either one.

The great mathematician Paul Erdős was fascinated by the difficulty of finding Ramsey numbers. Here's what he had to say:

"Imagine an alien force vastly more powerful than us landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet.

In that case, we should marshal all our computers and all our mathematicians and attempt to find the value.

But suppose, instead, that they ask for $R(6, 6)$.

Then we should attempt to destroy the aliens."

We can all do math every day!

NCTM

Texas Instruments

http://www.cbs.com/primetime/numb3rs/ti/activities.shtml