7.5: Non-homogeneous Recurrence Relations.

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k} = b$$

Let $k(h) = h_n - a_1 h_{n-1} - a_2 h_{n-2} - ... - a_k h_{n-k}$

Suppose ϕ is a solution to the recurrence relation $k(h) = 0$
and β is a solution to the recurrence relation $k(h) = b$.

Claim: $\phi + \beta$ is a solution to

To solve a non-homogeneous recurrence relation.

Step 1: Solve homogeneous equation.

Recall if constant coeffficents, guess $h_n = q^n$ for homogeneous eq'n.

Step 2: Guess a solution to non-homogeneous equation,
by guessing a solution β_n similar to $b(n)$.

Step 3a: Note general solution is $\sum c_i \phi_i(n) + \beta(n)$.

Step 3b: Find c_i using initial conditions.

Ex: Solve the recurrence relation: $h_n + h_{n-2} = 14n$, $h_0 = 3, h_1 = 5$

Step 1: Guess q^n is a solution to homogeneous equation:

$$h_n + h_{n-2} = 0.$$

$q^n + q^{n-2} = q^{n-2}(q^2 + 1) = 0$ $\qquad$ $q^2 + 1 = 0$ implies $q = \pm i$

Thus the general solution to homogeneous equation is

$$h_n = c_1 i^n + c_2 (-i)^n$$

Step 2: Guess a solution to non-homogeneous equation:

$$h_n + h_{n-2} = 14n$$

Guess $\beta_n =$

Plug β_n into non-homogeneous equation:

Solve for x and y: $2xn + 2y - 2x = 14n$ implies $x = 7$ and $y = 7$.

Thus a solution to non-homogeneous equation is $\beta(n) =$.

Step 3a: Note general soln to non-homogeneous equation is

Step 3b: Find c_i using initial conditions.

$$h_n + h_{n-2} = 14n,\ h_0 = 3, h_1 = 5$$

$h_0 = 3$: $c_1 i^0 + c_2(-i)^0 + 7(0) + 7 = 3$ implies $c_1 + c_2 = -4$

$h_1 = 5$: $c_1 i^1 + c_2(-i)^1 + 7(1) + 7 = 5$ implies $ic_1 - ic_2 = -9$

$c_1 + c_2 = -4$
$-c_1 + c_2 = -9i$ implies $c_1 = \frac{-4+9i}{2} = -2 + \frac{9i}{2}$ and $c_2 = \frac{-4-9i}{2} = -2 - \frac{9i}{2}$

$h_n = (-2 + \frac{9i}{2})i^n + (-2 - \frac{9i}{2})(-i)^n + 7n + 7$

$= (i^n)[(-2)(1 + (-1)^n) + (\frac{9i}{2})(1 - (-1)^n)] + 7n + 7$

$h_{2j} = (-1)^j(-4) + 7(2j) + 7 = 4(-1)^{j+1} + 7 + 14j$

$h_{2j+1} = (i^{2j+1})9i + 7(2j+1) + 7 = (i^{2j+2})9 + 14j + 14 = 9(-1)^{j+1} + 14j + 14$

Thus the sequence is 3, 5, 25, 37, 31, 33, 53, 65, 59, 61, 81, 93, ...