7.1: Sequences

**Arithmetic sequence:** $h_0, h_0+q, h_0+2q, ...$

$$h_n = h_{n-1} + q = h_0 + nq,\ n \geq 0$$

Example: $h_n = 3 + 5n$: 3, 8, 13, 18, 23, 28, ...

**Geometric sequence:** $h_0, qh_0, q^2h_0, ...$

$$h_n = qh_{n-1} = q^n h_0,\ n \geq 0$$

Example: $h_n = 2^n$: $1, 2, 4, 8, 16, 32, 62, 128, 256, 512, ...$

$h_n = 2^n$ = number of combinations of an $n$-element set.

**Partial sums:** $s_n = \sum_{k=0}^{n} h_k$

Partial sums of arithmetic sequence:

$$s_n = \sum_{k=0}^{n} h_0 + kq = \sum_{k=0}^{n} h_0 + \sum_{k=0}^{n} kq = (n+1)h_0 + \frac{qn(n+1)}{2}$$

Example: If $h_k = 3+5k$, then $s_n = \sum_{k=0}^{n} h_k = (n+1)3 + \frac{5n(n+1)}{2}$

3, 11, 24, 42, 65, 93, ....

Geometric sequence: $s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} \frac{q^{n+1}-1}{q-1} h_0 & q \neq 1 \\ (n+1)h_0 & q = 1 \end{cases}$

Example: If $h_k = 2^k$, then $s_n = \sum_{k=0}^{n} h_k = \frac{2^{n+1}-1}{2-1}$

1, 3, 7, 15, 31, 63, ....

**Fibonacci:**

Suppose a pair of rabbits of the opposite sex give birth to a pair of rabbits of opposite sex every month starting with their second month. If we begin with a pair of newly born rabbits, how many rabbits are there after one year.

Let $f_n = \#$ of pairs of rabbits at the beginning of month $n$

$f_0 =$ $\quad$ $f_1 =$ $\quad$ $f_2 =$ $\quad$ $f_3 =$ $\quad$ $f_4 =$ $\quad$ $f_5 =$

Hence $f_n =$

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Lemma: $s_n = \sum_{k=0}^{n} f_n = f_{n-2} - 1$

Proof by induction on $n$.

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Lemma: $f_n$ is even iff $3|n$.

Proof by induction on $n$.

Note that $f_0 = 0$ is even, $f_1 = 1$ is odd, and $f_2 = 1$ is odd.

Suppose $f_{3n}$ is even, $f_{3n+1}$ is odd, and $f_{3n+2}$ is odd.

Then $f_{3n+3} = f_{3n+2} + f_{3n+1}$. Since odd + odd is even, $f_{3n+3}$ is even.

Then $f_{3n+4} = f_{3n+3} + f_{3n+2}$. Since even + odd is odd, $f_{3n+4}$ is odd.

Then $f_{3n+5} = f_{3n+4} + f_{3n+3}$. Since odd + even is odd, $f_{3n+5}$ is odd.

Thm 7.1.2: $f_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

Proof: Check if $g(n) = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

satisfies $g(n) = g(n-1) + g(n-2)$ and $g(1) = 1$ and $g(2) = 1$

$$g(1) = \sum_{k=0}^{1-1} \binom{1-1-k}{k} = \sum_{k=0}^{0} \binom{-k}{k} = \binom{0}{0} = 1$$

$$g(2) = \sum_{k=0}^{2-1} \binom{2-1-k}{k} = \sum_{k=0}^{1} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1$$

$$g(n-1) + g(n-2)$$

$$= \sum_{k=0}^{n-1-1} \binom{n-1-1-k}{k} + \sum_{k=0}^{n-2-1} \binom{n-2-1-k}{k}$$

$$= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{k=0}^{n-3} \binom{n-3-k}{k}$$

$$= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-(k-1)}{k-1}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-3-(k-1)}{k-1}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-2-k}{k-1}$$

$$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \left[ \binom{n-2-k}{k} + \binom{n-2-k}{k-1} \right]$$

$$= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k}$$

$$= \binom{n-2}{0} + \sum_{k=0}^{n-1} \binom{n-1-k}{k} - \binom{n-1}{0} - \binom{0}{n-1}$$

$$= 1 + \sum_{k=0}^{n-1} \binom{n-1-k}{k} - 1 - 0$$

$$= \sum_{k=0}^{n-1} \binom{n-1-k}{k}$$

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Fibonacci sequence is defined by

Homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$

and initial conditions: $f(0) = 0$, $f(1) = 1$.

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Thm 7.1.1: $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: Suppose $f_n = x^n$. Then $f_{n-1} = x^{n-1}$ and $f_{n-2} = x^{n-2}$

Then $0 = f_n - f_{n-1} - f_{n-2} = x^n - x^{n-1} - x^{n-2}$

Thus $x^{n-2}(x^2 - x - 1) = 0$.

Thus either $x = 0$ or $x = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Thus $f_n = 0$, $\quad f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Hence $f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies the

homogeneous linear recurrence relation: $f_n - f_{n-1} - f_{n-2} = 0$.

Suppose the initial conditions are $f_0 = a$ and $f_1 = b$

(note for fibonacci sequence, $a = 0$ and $b = 1$).

Then for $n = 0$: $f_0 = c_1 + c_2 = a$

And for $n = 1$: $f_1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = b$

Or in matrix form: $\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{-2\sqrt{5}} a + \frac{b}{\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} a - \frac{b}{\sqrt{5}} \end{pmatrix}$$

If $a = 0$ and $b = 1$, then $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

5.6

$\frac{1}{(1-x)^n} = \Sigma_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \Sigma_{k=0}^{\infty} \binom{n+k-1}{k} x^k$, $|x| < 1$

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$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$

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$\frac{x^{n+1}-1}{x-1} = 1 + x + x^2 + x^3 + ... + x^n$

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7.2: Generating Functions

$g(x) = h_0 + h_1 x + h_2 x^2 + ....$ is the *generating function* for the sequence $h_0, h_1, h_2, .....$

Ex: The generating fn for the sequence 2, 3, 4, 0, 0, 0, ... is

$$g(x) = 2 + 3x + 4x^2$$

Ex: The generating function for the sequence 1, 1, 1, ... is

$$g(x) = 1 + x + x^2 + ... = \frac{1}{1-x}$$

Ex: The generating function for the sequence
0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, ... is

$$g(x) = x^4 + x^7 + x^{10} + ... = x^4(1 + x^3 + x^6 + ...) = \frac{x^4}{1-x^3}$$

Ex: The generating function for the sequence

$\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, ..., \binom{m}{m}$ is

$$g(x) = \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + ... \binom{m}{m} x^m = (1+x)^m$$

Ex: Suppose $\alpha \in \mathcal{R}$. The generating function for the sequence

$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, ...$ is

$$g(x) = \binom{\alpha}{0} + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + ... = (1+x)^\alpha$$

Ex: Let $h_n$ = number of nonnegative solutions to

$$e_1 + e_2 + ... + e_k = n$$

Thus $h_n =$

Thus $g(x) =$

Suppose a multiset of size $k$ must contain the following:

between two to four (inclusive) $x$'s,
zero, one, two or five $y$'s.

Find the number of multisets of size $k$.

**"Long" method: list all possibilities**

between two to four (inclusive) $x$'s: $x^2 + x^3 + x^4$

zero, one, two or five $y$'s: $y^0 + y^1 + y^2 + y^5$

Both: $(x^2 + x^3 + x^4)(y^0 + y^1 + y^2 + y^5)$

$$= x^2y^0 + x^2y^1 + x^2y^2 + x^2y^5 + x^3y^0 + x^3y^1 + x^3y^2 + x^3y^5 + x^4y^0 + x^4y^1 + x^4y^2 + x^4y^5$$

$$= x^2y^0 + (x^2y^1 + x^3y^0) + (x^2y^2 + x^3y^1 + x^4y^0) + (x^3y^2 + x^4y^1) + x^4y^2 + x^2y^5 + x^3y^5 + x^4y^5$$

Let $h_k$ = number of multisets of size $k$.

$h_0 =$ , $h_1 =$ , $h_2 =$ , $h_3 =$ , $h_4 =$ ,
$h_5 =$ , $h_6 =$ , $h_7 =$ , $h_8 =$ , $h_9 =$ ,
$h_k =$ $k > 9$

**"Shorter" method:**

between two to four (inclusive) $x$'s: $x^2 + x^3 + x^4$

zero, one, two or five $y$'s: $x^0 + x^1 + x^2 + x^5$

Both: $g(x) = (x^2 + x^3 + x^4)(x^0 + x^1 + x^2 + x^5)$

$$= x^2x^0 + (x^2x^1 + x^3x^0) + (x^2x^2 + x^3x^1 + x^4x^0) + (x^3x^2 + x^4x^1) + x^4x^2 + x^2x^5 + x^3x^5 + x^4x^5$$

$$= x^2 + 2x^3 + 3x^4 + 2x^5 + x^6 + x^7 + x^8 + x^9$$

Suppose a multiset consisting of integers between 0 and 5 inclusive of size $k$ must contain the following:

even number of 0's: $x^0 + x^2 + x^4 + ... = \frac{1}{1-x^2}$

odd number of 1's: $x^1 + x^3 + x^5 + ... = \frac{x}{1-x^2}$

three or four 2's: $x^3 + x^4 = x^3(1+x)$

the number of 3's is a multiple of five: $x^0 + x^5 + x^{10} + ... = \frac{1}{1-x^5}$

btwn zero to four (inclusive) 4's: $x^0 + x^1 + x^2 + + x^3 + x^4 = \frac{1-x^5}{1-x}$

zero or one 5: $x^0 + x^1 = 1 + x$

$$g(x) = (x^0 + x^2 + x^4 + ...)(x^1 + x^3 + x^5 + ...)(x^3 + x^4)$$
$$(x^0 + x^5 + x^{10} + ...)(x^0 + x^1 + x^2 + + x^3 + x^4)(x^0 + x)$$

$$= \left(\frac{1}{1-x^2}\right)\left(\frac{x}{1-x^2}\right)x^3(1+x)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x)$$

$$= \frac{x^4}{(1-x)^3} = x^4 \Sigma_{k=0}^{\infty} \binom{3+k-1}{k} x^k = \Sigma_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^{k+4}$$

Find the number of multisets of size $n$.

Find the number of multisets of size 100.

Determine the generating function for $h_n$ = the number of ways to make $n$ cents using pennies, nickels, dimes, and quarters.

Note $h_n$ = the number of nonnegative integral solutions to

$$e_1 + 5e_2 + 10e_3 + 25e_4 = n$$

Let $f_1 = e_1,\ f_2 = 5e_2,\ f_3 = 10e_3,\ f_4 = 25e_4,$

Then $h_n$ = the number of nonnegative integral solutions to $f_1 + f_2 + f_3 + f_4 = n$

where $f_1$ is a nonnegative integer

$f_2$ is a multiple of 5

$f_3$ is a multiple of 10

$f_4$ is a multiple of 25

Hence the generating function for $h_n$ is