7.1: Sequences

**Arithmetic sequence:** $h_0, h_0+q, h_0+2q, ...$

$$h_n = h_{n-1} + q = h_0 + nq,\ n \geq 0$$

Example: $h_n = 3 + 5n$: 3, 8, 13, 18, 23, 28, ...

**Geometric sequence:** $h_0, qh_0, q^2h_0, ...$

$$h_n = qh_{n-1} = q^n h_0,\ n \geq 0$$

Example: $h_n = 2^n$: $1, 2, 4, 8, 16, 32, 62, 128, 256, 512, ...$

$h_n = 2^n =$ number of combinations of an $n$-element set.

**Partial sums:** $s_n = \sum_{k=0}^{n} h_k$

Partial sums of arithmetic sequence:

$$s_n = \sum_{k=0}^{n} h_0 + kq = \sum_{k=0}^{n} h_0 + \sum_{k=0}^{n} kq = (n+1)h_0 + \frac{qn(n+1)}{2}$$

Example: If $h_k = 3+5k$, then $s_n = \sum_{k=0}^{n} h_k = (n+1)3 + \frac{5n(n+1)}{2}$

3, 11, 24, 42, 65, 93, ....

Geometric sequence: $s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} \frac{q^{n+1}-1}{q-1} h_0 & q \neq 1 \\ (n+1)h_0 & q = 1 \end{cases}$

Example: If $h_k = 2^k$, then $s_n = \sum_{k=0}^{n} h_k = \frac{2^{n+1}-1}{2-1}$

1, 3, 7, 15, 31, 63, ....

**Fibonacci:**

Suppose a pair of rabbits of the opposite sex give birth to a pair of rabbits of opposite sex every month starting with their second month. If we begin with a pair of newly born rabbits, how many rabbits are there after one year.

Let $f_n = \#$ of pairs of rabbits at the beginning of month $n$

$f_0 =$ $f_1 =$ $f_2 =$ $f_3 =$ $f_4 =$ $f_5 =$

Hence $f_n =$

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Lemma: $s_n = \sum_{k=0}^{n} f_n = f_{n-2} - 1$

Proof by induction on $n$.

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Lemma: $f_n$ is even iff $3|n$.

Proof by induction on $n$.

Note that $f_0 = 0$ is even, $f_1 = 1$ is odd, and $f_2 = 1$ is odd.

Suppose $f_{3n}$ is even, $f_{3n+1}$ is odd, and $f_{3n+2}$ is odd.

Then $f_{3n+3} = f_{3n+2} + f_{3n+1}$. Since odd + odd is even, $f_{3n+3}$ is even.

Then $f_{3n+4} = f_{3n+3} + f_{3n+2}$. Since even + odd is odd, $f_{3n+4}$ is odd.

Then $f_{3n+5} = f_{3n+4} + f_{3n+3}$. Since odd + even is odd, $f_{3n+5}$ is odd.

Thm 7.1.2: $f_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

Proof: Check if $g(n) = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$

satisfies $g(n) = g(n-1) + g(n-2)$ and $g(1) = 1$ and $g(2) = 1$

Thm 7.1.1: $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$