6.6 Mobius inversions

Defn: The Euler function $\phi(n) = |S_n|$
where $S_n = \{k \ : \ 1 \leq k \leq n, GCD(k, n) = 1\}$

Ex: $\phi(1) =$ ________, $\phi(6) =$ ________, $\phi(15) =$ ________

$\mathcal{N}$ is partially ordered by $k \leq n$ iff $k|n$

Let $F : \mathcal{N} \to \mathcal{R}$, $F(n) = \phi(n)$

Then $G(n) =$

By thm 6.6.1 $\phi(n) =$

Ex: $G(6) = \phi(1) + \phi(2) + \phi(3) + \phi(6) = 1 + 1 + 2 + 2 = 6$

Note that $G(d) = d$.

Hence $\phi(n) = \Sigma_{\{d \ : \ d|n\}} d \ \mu(d, n)$.

Let $\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

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The Mobius function, $\mu = \zeta^{-1}$

Suppose $f(x,x) \neq 0 \forall x \in X$

Then $f^{-1}(x,x) = \frac{1}{f(x,x)}$

$$f^{-1}(x,y) = -\Sigma_{\{z\,:\,x \leq z < y\}} f^{-1}(x,z) \frac{f(z,y)}{f(y,y)} \text{ for } x < y,$$

$$f^{-1}(x,y) = 0 \text{ for } x > y.$$

Since $\zeta(x,x) = 1 \forall x \in X$, Mobius fn, $\mu = \zeta^{-1}$ exists.

$$\mu(x,x) = \frac{1}{\zeta(x,x)} = 1$$

Hence $\mu(1,1) = 1$

$$\mu(x,y) = -\Sigma_{\{z\,:\,x \leq z < y\}} \mu(x,z) \frac{\zeta(z,y)}{\zeta(y,y)} = -\Sigma_{\{z\,:\,x \leq z < y\}} \mu(x,y)$$

$$\mu(1,d) = -\Sigma_{\{z\,:\,1 \leq z < d\}} \mu(1,z) = -\Sigma_{\{z\,:\,z | d, z \neq d\}} \mu(1,z)$$

$$\mu(1,2) = -\Sigma_{\{z\,:\,1 \leq z < 2\}} \mu(1,z) = -\mu(1,1) = -1$$

$$\mu(1,3) = -\Sigma_{\{z\,:\,1 \leq z < 3\}} \mu(1,z) = -\mu(1,1) = -1$$

$$\mu(1,4) = -\Sigma_{\{z\,:\,1 \leq z < 4\}} \mu(1,z) =$$

$$\mu(1,5) = -\Sigma_{\{z\,:\,1 \leq z < 5\}} \mu(1,z) = -\mu(1,1) = -1$$

$$\mu(1,6) = -\Sigma_{\{z\,:\,1 \leq z < 6\}} \mu(1,z) = -\mu(1,1) - \mu(1,2) - \mu(1,3) =$$

$$\mu(1,12) = -\Sigma_{\{z\,:\,1 \leq z < 30\}} \mu(1,z) = -\mu(1,1) - \mu(1,2) - \mu(1,3) - \mu(1,4) - \mu(1,6) =$$

$$\mu(1,30) = -\Sigma_{\{z\,:\,1 \leq z < 30\}} \mu(1,z) = -\mu(1,1) - \mu(1,2) - \mu(1,3) - \mu(1,5) - \mu(1,6) - \mu(1,10) - \mu(1,15) =$$

Suppose $d = p_1^{k_1} p_2^{k_2} .... p_r^{k_r}$

$$\mu(1,d) = \begin{cases} 1 & \text{if } d = 1 \\ (-1)^r & \text{if } k_i = 1 \forall i \\ 0 & \text{otherwise} \end{cases}$$

Suppose $kx = y$, then $\mu(x,y) = \mu(1, y/x)$

Hence $\phi(n) = \Sigma_{\{d \ : \ d|n\}} d \ \mu(d,n) =$

Let $S_n^d = \{k \ : \ 1 \leq k \leq n, GCD(k,n) = d\}$

Lemma: If $d|n$, then $|S_n^d| = \phi(n/d)$

Proof: Let $f : S_{\frac{n}{d}} \rightarrow S_n^d$, $f(k) = dk$

Claim: f is well defined:

Claim f is onto:

Hence f is a bijection and $|S_n^d| = \phi(n/d)$