6.6 Mobius inversions

Let $X_n = \{1, 2, ..., n\}$

$\mathcal{P}(X_n) = \{A \subset X_n\}$

$\mathcal{P}(X_n)$ is partially ordered by the relation $\subset$.

Suppose $F : \mathcal{P}(X_n) \to \mathcal{R}$

Define $G : \mathcal{P}(X_n) \to \mathcal{R}$ by $G(K) = \Sigma_{L \subset K} F(L)$

Claim: we can invert this equation to recover F from G:

$$F(K) = \Sigma_{L \subset K} (-1)^{|K|-|L|} G(L)$$

Example: Let S be any finite set. Let $A_i \subset S$ and
let A_i be indexed by elements of X_n
(ie, we have A_1, A_2, ..., A_n).

Let $F : \mathcal{P}(X_n) \to \mathcal{R}$ be defined by
$F(K) = |\{s \mid s \notin A_i \forall i \in K, s \in A_j \forall j \notin K\}|$

$\{s \mid s \notin A_i \ \forall i \in K, s \in A_j \ \forall j \notin K\}$
$= \{s \mid s \in \overline{A_i} \ \forall i \in K, s \in A_j \ \forall j \in \overline{K}\}$
$= (\cap_{i \in K} \overline{A_i}) \cap (\cap_{i \in \overline{K}} A_i)$

Suppose $S = \{a_1, a_2, a_3, a_4\}$. Suppose $X_n = \{1, 2\}$

Let $A_1 = \{a_1, a_2, a_3\}$. Let $A_2 = \{a_1, a_2\}$.

$F(\emptyset) = |(\cap_{i \in \emptyset} \overline{A_i}) \cap (\cap_{i \in X_n} A_i)| = |A_1 \cap A_2| = |\{a_1, a_2\}| = 2$

$F(\{1\}) = |(\cap_{i \in \{1\}} \overline{A_i}) \cap (\cap_{i \in \{2\}} A_i)| = |\overline{A_1} \cap A_2| = |\emptyset| = 0$

$F(\{2\}) = |(\cap_{i \in \{2\}} \overline{A_i}) \cap (\cap_{i \in \{1\}} A_i)| = |\overline{A_2} \cap A_1| = |\{a_3\}| = 1$

$F(\{1, 2\}) = |(\cap_{i \in \{1,2\}} \overline{A_i}) \cap (\cap_{i \in \emptyset} A_i)| = |\overline{A_1} \cap \overline{A_2}| = |\{a_4\}| = 1$

$G(K) = \Sigma_{L \subset K} F(L) = | \cap_{i \in \overline{K}} A_i|$

$G(\emptyset) = F(\emptyset) = 2$

$G(\{1\}) = F(\emptyset) + F(\{1\}) = 2$

$G(\{2\}) = F(\emptyset) + F(\{2\}) = 3$

$G(\{1, 2\}) = F(\emptyset) + F(\{1\}) + F(\{2\}) + F(\{1, 2\}) = 4$

By claim, $F(K) = \Sigma_{L \subset K} (-1)^{|K|-|L|} G(L)$

Hence $F(X_n) = \Sigma_{L \subset X_n} (-1)^{|X_n|-|L|} G(L)$

$$= \Sigma_{L \subset X_n} (-1)^{|\overline{L}|} | \cap_{i \in \overline{L}} A_i|$$

$$= \Sigma_{\overline{L} \subset X_n} (-1)^{|\overline{L}|} | \cap_{i \in \overline{L}} A_i|$$

$$= \Sigma_{K \subset X_n} (-1)^{|K|} | \cap_{i \in K} A_i|$$