6.6 Mobius inversions

Let $X$ be a finite set.

Let $\mathcal{F} = \{f : X \times X \to \mathcal{R} \mid \text{ if } f(x, y) \neq 0, \text{ then } x \leq y\}$

Define the operation * on $\mathcal{F}$ by

$$f * g = \begin{cases} \Sigma_{\{z \ \mid \ x \leq z \leq y\}} f(x, z) g(z, y) & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Note $*$ is associative: $f * (g * h) = (f * g) * h$

For example $\delta = \begin{cases} 1 & \text{x = y} \\ 0 & \text{otherwise} \end{cases}$

Note $\delta$ acts as the identity for *: $f * \delta = \delta * f = f$

Let $\zeta(x, y) = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise} \end{cases}$

If $f(x, x) \neq 0$ for all $x \in X$, then $f$ is invertible: There exist $f^{-1}$ such that $f * f^{-1} = f^{-1} * f = \delta$.

Def. The Mobius function, $\mu = \zeta^{-1}$

Example: if $X = X_n = \{1, 2, ..., n\}$ and $\mathcal{P}(X_n)$ is partially ordered by the relation $\subset$, then $\mu(A, B) = (-1)^{|B| - |A|}$

Thm 6.6.1. Let $(X, \leq)$ be a partially ordered set with a smallest element 0. If $F : X \to \mathcal{R}$, define $G : X \to \mathcal{R}$ by $G(x) = \Sigma_{\{z \ \mid \ z \leq x\}} F(z)$. Then

$F(x) = \Sigma_{\{y \ \mid \ y \leq x\}} G(y) \mu(y, x)$

Proof: Evaluate $\Sigma_{\{y \ \mid \ y \leq x\}} G(y) \mu(y, x)$