6.4 Permutations with Forbidden Positions

Goal: To **derive** a more general formula for counting the number of permutations with arbitrary forbidden positions.

Recall in section 6.3, we looked at permutations with forbidden positions A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$. In this section, we will cover arbitrary forbidden positions.

Let $X_j \subset \{1, ..., n\}$ for $j = 1, ..., n$.

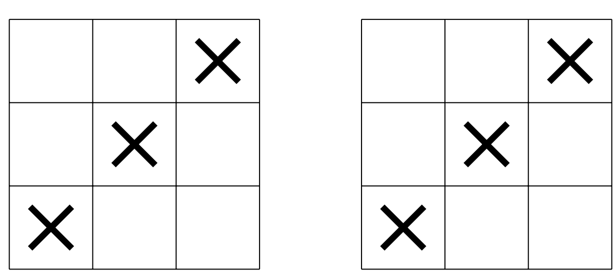
Defn: $P(X_1, X_2, ..., X_n)$ = the set of permutations $i_1 i_2 ... i_n$ of $\{1, ..., n\}$ such that $i_j \notin X_j$.

Defn: $p(X_1, X_2, ..., X_n) = |P(X_1, X_2, ..., X_n)|$

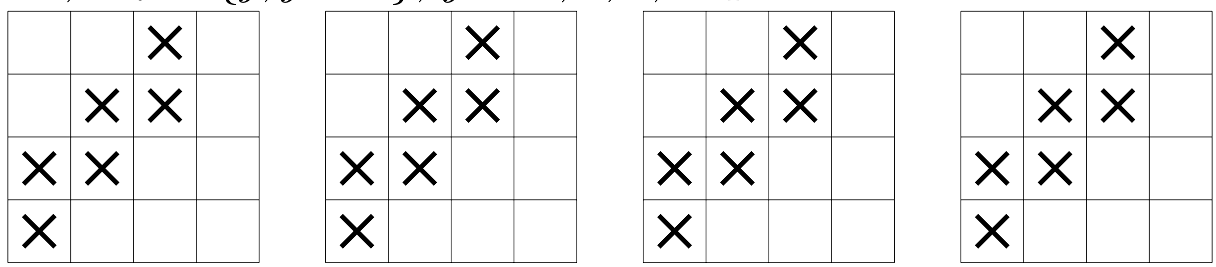
Ex: $P(X_1, X_2, ..., X_n)$ corresponds to the set of derangements of $\{1, ..., n\}$ if $X_j = \{j\}$. Thus $D_n = |P(\{1\}, \{2\}, ..., \{n\}|$

Recall, we can visualize permutations with forbidden positions via $n \times n$ chessboards.

Ex: Derangements of $\{1, 2, 3\}$:
$X_j = \{j\}$.



Non-derangement example:
$n = 4,\ X_i = \{j, j+1\},\ j = 1, 2, 3,\ X_4 = \emptyset$.



$$P(X_1, X_2, ..., X_n) = P(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= \{3124, 3412, 3421, 4123\}.$$

$$p(X_1, X_2, ..., X_n) = p(\{1,2\}, \{2,3\}, \{3,4\}, \emptyset)$$
$$= |\{3124, 3412, 3421, 4123\}| = 4.$$

We can use the inclusion-exclusion principle to calculate $p(X_1, X_2, ..., X_n)$ (although in many cases, the computation can be tediously long and beyond computer capabilities for large n).

Thm 6.4.1:
$p(X_1, X_2, ..., X_n) = n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n.$

Proof (Similar to the proof of Thm 6.3.1.):

By the inclusion-exclusion principle,

$$p(X_1, X_2, ..., X_n) =$$
$$|S| - \Sigma_{j=1}^{n} |A_j| + \sum_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$$

where

Let S = the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Let A_j = set of permutations $i_1 i_2 ... i_n$ such that $i_j \in X_j$ (for a fixed j).

Note there are $|X_j|$ ways to place a rook in the jth position.

There are $(n-1)!$ ways to place the remaining $n-1$ rooks so that the permutation belongs to A_j.

Thus $|A_j| = |X_j|(n-1)!$.

$\Sigma_{j=1}^{n}|A_j| = \Sigma_{j=1}^{n}|X_j|(n-1)! = (n-1)!\Sigma_{j=1}^{n}|X_j| = r_1(n-1)!$

where $r_1 = \Sigma_{j=1}^{n}|X_j|$.

Note r_1 = number of ways to place 1 nonattacking rooks on an $n \times n$ chessboard so that the rook is in a forbidden position.

Let's now look at $A_j \cap A_k$.
$i_1 i_2 ... i_n \in A_j \cap A_k$, then $i_j \in X_j$ and $i_k \in X_k$.

There are $|X_j|$ ways to place a rook in the jth position and $|X_k|$ ways to place a rook in the kth position in forbidden positions IF we do not restrict to non-attacking positions. But since we require non-attacking positions, placing the two rooks in the ith and jth columns are not independent events.

Let $\rho_{i,j}$ = number of ways to place 2 nonattacking rooks in columns i and j on an $n \times n$ chessboard so that each of the 2 rooks is in a forbidden position.

There are $(n-2)!$ ways to place the remaining $n-2$ rooks so that the permutation belongs to $A_j \cap A_k$.

Thus $|A_j \cap A_k| = (\rho_{i,j})(n-2)!$

$$\sum_{i,j} |A_i \cap A_j| = \sum_{i,j} (\rho_{i,j})(n-2)! = (n-2)! \sum_{i,j} \rho_{i,j} = r_2(n-2)!$$

where r_2 = number of ways to place 2 nonattacking rooks on an $n \times n$ chessboard so that each of the 2 rooks is in a forbidden position.

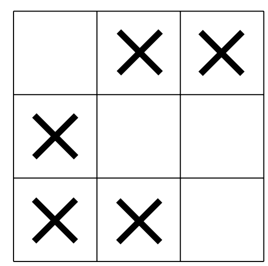
Similarly, define r_k = number of ways to place k nonattacking rooks on an $n \times n$ chessboard so that each of the k rooks is in a forbidden position.

Then $\Sigma|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = r_k(n-k)!$. □

Note that if there are many forbidden positions, then r_k may be difficult to calculate and it may be easier to calculate $p(X_1, X_2, ..., X_n)$ directly. If there are few forbidden positions, Thm 6.4.1 is the easier method to compute $p(X_1, X_2, ..., X_n)$.

Example: Let $X = \{1, 2, 3\}$. Let $i_1 i_2 i_3 \in P(\{1, 2\}, \{1, 3\}, \{3\})$. Then there is only one choice for both i_1 and i_2, namely $i_1 = 3$ and $i_2 = 2$. But this leaves only one choice for $i_3 = 1$.

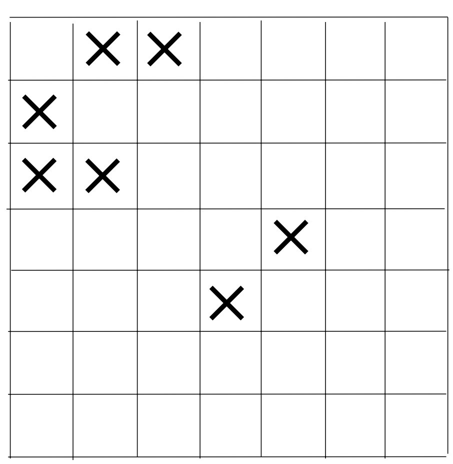
Thus $p(\{1, 2\}, \{1, 3\}, \{3\}) = 1$.



Note in this case, it was easiest to count directly and not use Thm 6.4.1.

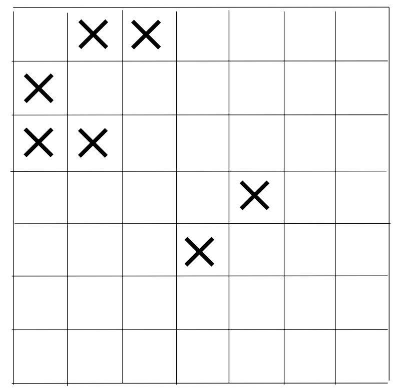
Example: Let $X = \{1, 2, 3, 4, 5, 6, 7\}$.

Calculate $p(\{1, 2\}, \{1, 3\}, \{3\}, \{5\}, \{4\}, \emptyset, \emptyset)$



r_1 = the number of forbidden positions = 7.

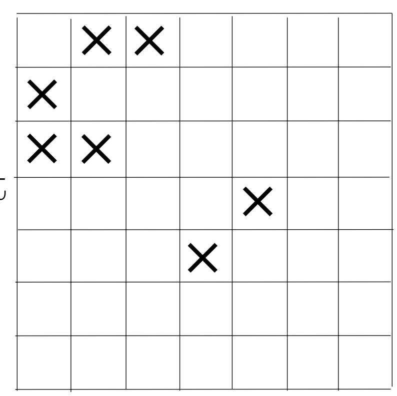
To calculate r_i for $i > 1$, note that the forbidden positions can be partitioned into two independent sets. Let F_1 = the forbidden positions in the upper left 3×3 corner. Let F_2 contain the other two forbidden positions in a 2×2 square. These positions are independent because a rook in F_1 cannot attack a rook in F_2 (and vice versa).



To calculate r_2 we break the problem into the following cases:

Case 1: both rooks are in F_1.

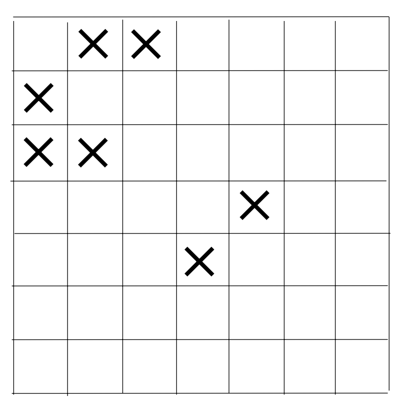
Subcase 1: one rook is placed in the 3rd column. There is only 1 possible placement for a rook in the 3rd column (1st row). There are 3 possible placements for the second rook so that the two rooks are in F_1 in non-attacking position.



Subcase 2: both rooks are place in the first two columns. In this case, the placement possibilities correspond to 13, 21, 23. Thus there are 3 possibilities in this subcase.

Thus there are $3 + 3 = 6$ possible non-attacking rook placements when both rooks are in F_1.

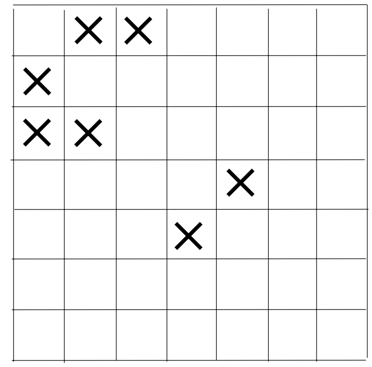
Case 2: both rooks are in F_2.



Since there are only 2 rooks in F_2, there is only one way to place both rooks in F_2.

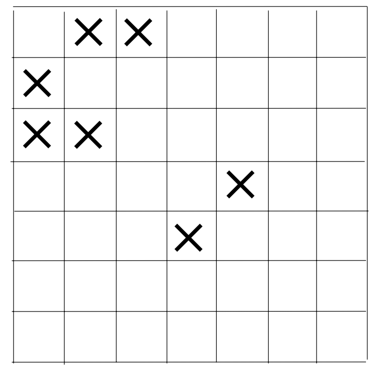
Case 3: one rook is in F_1 while the other rook is in F_2.

There are $(5)(2) = 10$ ways to place one rook in F_1 and one rook in F_2.



Thus $r_2 = 6 + 1 + 10 = 17.$

To calculate r_3 we break the problem into the following cases:



Case 1: all 3 rooks are in F_1:

There is only 1 possible placement for a rook in the 3rd column (1st row). Thus in the second column, the rook must be placed in the first row. Thus in the first column, the remaining rook must be placed in the second row

Thus the only valid placement corresponds to the permutation 213

Case 2: 2 rooks in F_1, 1 rook in F_2.

By above, there are 6 ways to place 2 rooks in F_1 and 2 ways to place 1 rook in F_2. Thus there are $(6)(2) = 12$ ways to place 2 rooks in F_1, 1 rook in F_2.

Case 3: 1 rook in F_1, 2 rooks in F_2.

By above, there are 5 ways to place 1 rook in F_1 and 1 way to place 2 rook in F_2. Thus there are $(5)(1) = 5$ ways to place 1 rooks in F_1, 2 rook in F_2.

Hence $r_3 = 1 + 12 + 5 = 18$

To calculate r_4 we break the problem into the following cases:

Case 1: 3 rooks in F_1, 1 rook in F_2:

By above, there is 1 way to place 3 rooks in F_1 and 2 ways to place 1 rook in F_2. Thus there are $(1)(2) = 2$ ways to place 3 rooks in F_1, 1 rook in F_2.

Case 2: 2 rooks in F_1, 2 rook in F_2.

By above, there are 6 ways to place 2 rooks in F_1 and 1 way to place 1 rook in F_2. Thus there are $(6)(1) = 6$ ways to place 2 rooks in F_1, 2 rook in F_2.

Hence $r_4 = 2 + 6 = 8$

To calculate r_5 we note that if we have 5 nonattacking rooks in forbidden positions, 3 rooks are in F_1 and 2 rook in F_2. By above, there is 1 way to place 3 rooks in F_1 and 1 way to place 2 rook in F_2. Thus there are $(1)(1) = 1$ way to place 3 rooks in F_1, 2 rook in F_2. **Thus** $r_5 = 1$.

Note $r_6 = r_7 = 0$ as we can't place more than 5 nonattacking rooks in forbidden positions if we only have 5 columns which contain forbidden positions.

Hence $p(\{1,2\},\{1,3\},\{3\},\{5\},\{4\},\emptyset,\emptyset)$

$$= n! - r_1(n-1)! + r_2(n-2)! - ... + (-1)^n r_n$$

$$= 7! - r_1(6!) + r_2(5!) - r_3(4!) + r_4(3!) - r_5(2!)$$

$$= 7! - 7(6!) + 17(5!) - 18(4!) + 8(3!) - (2!).$$