6.3 Derangements

Suppose each person in a group of n friends brings a gift to a party. In how many ways can the n gifts be distributed so that each person receives one gift and no person receives their own gift.

Let the set of friends $= \{p_1, ..., p_n\}$ where $p_j =$ person j.
Let the set of gifts $= \{g_1, ..., g_n\}$ where $g_j =$ the gift brought by person j.

Suppose $f : \{p_1, ..., p_n\} \to \{g_1, ..., g_n\}$,
$f(p_k) = g_j$ iff person p_k receives give g_j, the gift brought by person j.
If each person receives one gift, then f is a bijection.
If no person receives their own gift. Then $f(p_j) \neq g_j$.

In simpler notation,
$f : \{1, ..., n\} \to \{1, ..., n\}$ such that $f(j) \neq j$

Recall:
a permutation on $\{1, ..., n\}$ is a bijection $f : \{1, ..., n\} \to \{1, ..., n\}$

Ex: The permutation 1 2 3 4 5 corresponds to the identity function.

Ex: The permutation 1 3 2 corresponds to the function
$f(1) = 1,\ f(2) = 3,\ f(3) = 2$

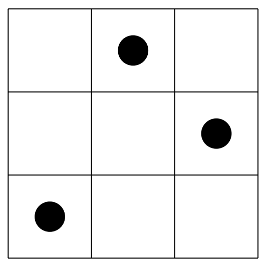
Defn: A *derangement* of $\{1, ..., n\}$ is a permutation $i_1 i_2 ... i_n$ such that $i_j \neq j$. I.e, j is not in the jth place.

In function notation:

$$f(j) = i_j, \text{ then if } i_1 i_2 ... i_n \text{ is a derangement, } f(j) \neq j.$$

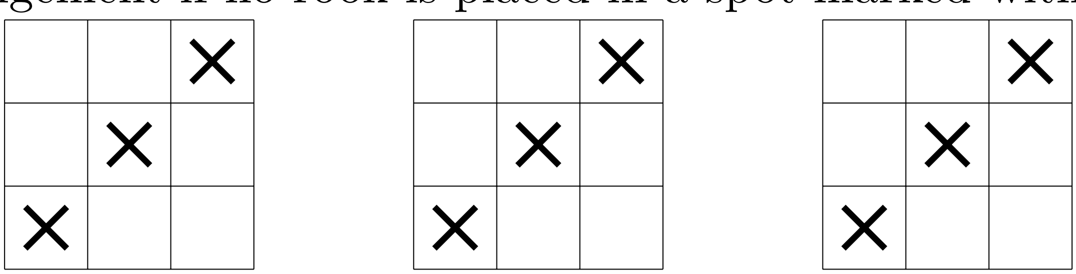
In yet other wording, recall a permutation corresponds to the placement of n non-attacking rooks on an $n \times n$ chessboard.

Ex: The permutation 1 3 2 corresponds to the following rook placement:



A derangement corresponds to non-attacking rook placement with forbidden positions along the diagonal (j, j), for $j = 1, ..., n$.

Ex: If rooks are placed on the following 3×3 chessboard in non-attacking position, then the rook placement corresponds to a derangement if no rook is placed in a spot marked with an X.



Thus the derangements of $\{1, 2, 3\}$ are 2 3 1 and 3 1 2.

Let D_n = the number of derangements of $\{1, ..., n\}$.

Thus $D_3 = 2$.

Thm 6.3.1: For $n \geq 1$, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!})$

Pf: Use the inclusion and exclusion principle: If $A_i \subset S$,
$\overline{\cup A_i} = |S| - \Sigma_{j=1}^n |A_j| + \Sigma_{i,j} |A_i \cap A_j| - ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$.

Choose S: What can we count which contains the set of derangements?

Let $S =$ the set of permutations of $\{1, ..., n\}$. Then $|S| = n!$.

Choose A_j such that the set of derangements $= \overline{\cup A_j}$.
Let $A_j =$ set of permutations such that j is in the jth spot.

$|A_j| = (n-1)!$ since there is only one choice for the jth spot (namely j), leaving $n-1$ terms to permute in the remaining $n-1$ places.

$|A_i \cap A_j| = (n-2)!$ since there is only one choice for the ith spot (namely i) and only one choice for the jth spot (namely j), leaving $n-2$ terms to permute in the remaining $n-2$ places.

Similarly, $|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}| = (n-k)!$.

Thus $D_n = n! - \Sigma_{j=1}^n (n-1)! + \Sigma_{i,j} (n-2)! - ... + (-1)^n (n-n)!$

$$= \binom{n}{0} n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - ... + \binom{n}{n} (-1)^n 0!$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} + ... + (-1)^n \frac{n!}{n!} = n!(1 - \frac{1}{1!} + \frac{1}{2!} + ... + (-1)^n \frac{1}{n!})$$

Recall $\binom{n}{k} =$ number of ways to choose k A_i's.

Sidenote: Finding the number of derangements is often called the hat check problem, because in the old days it was sometimes stated in the following terms: If n men check their hats, what is the probability that the hats are returned so that no one received their own hat.

Recall: If $E \subset S$, then the probability of $E = P(E) = \frac{|E|}{|S|}$

S = sample space, E = events.

Note: we assume each outcome is equally likely.

Suppose 4 customers at a restaurant order 4 meals. What is the probability that a waiter delivers these 4 orders to the 4 customers so that no customer receives what they ordered?

Answer: $\frac{D_4}{4!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = 0.375$

The probability that a permutation of $\{1, ..., n\}$ is a derangement $= \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} + ... + (-1)^n \frac{1}{n!}$

Recall Taylor's expansion from Calculus I,
$f(x) = \Sigma_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j$ (under appropriate hypothesis).

Thus $e^{-1} = \Sigma_{j=0}^{\infty}(-1)^j \frac{1}{j!}$ (let $f(x) = e^x, x = -1, a = 0$).

Thus e^{-1} is a good approximation for the probability of a derangement for n (slightly) large.

Thus the probability of a derangement is about the same when $n = 5$ as it is for $n = 50000000000$.

$\frac{D_5}{5!} = 0.3\overline{6}$, $\frac{D_6}{6!} = 0.3680\overline{5}$, $\frac{D_7}{7!} = 0.36785714285$, $e^{-1} = 0.36787944117...$

We can derive a recursive formula for D_n (we will look at many recursive formulas in chapter 7).

Lemma A: $D_n = (n-1)(D_{n-2} + D_{n-1})$ for $n \geq 3$.

Note the above formula is a recursive formula as we can determine D_n by calculating D_k for $k < n$.

Note $D_1 = 0$, $D_2 = 1$ (as 2 1 is the only derangement of $\{1, 2\}$).

Thus $D_3 = 2(0+1) = 2$, $D_4 = 3(1+2) = 9$, $D_5 = 4(2+9) = 44$, etc.

Combinatorial proof of lemma A:

Let $\mathcal{D}_n =$ the set of derangements of $\{1, ..., n\}$.

$D_n =$ the number of derangements of $\{1, ..., n\} = |\mathcal{D}_n|$.

We need to show that D_n is a product of $n-1$ and $D_{n-2}+D_{n-1}$. If we can partition $\mathcal{D}_n$ into $n-1$ subsets where each subset has $D_{n-2} + D_{n-1}$ elements, we can use the multiplication principle to show $D_n = (n-1)(D_{n-2} + D_{n-1})$.

We also want to relate D_n to $D_{n-1} =$ the number of dearrangements of $\{1, ..., n-1\}$ (and D_{n-2}).

Let's focus on one of the positions of a derangement. The last (nth) position of our derangement can be anything except n. Thus there are $n-1$ choices for the last (nth) position. Note the factor $n-1$ appears in our formula.

Let $\mathcal{R}_k =$ the set of derangements of $\{1, ..., n\}$ where k is in the nth position for $k = 1, ..., n-1$.

Then $\mathcal{D}_n = \cup_{j=0}^{n-1} \mathcal{R}_n$

Let $r_k = |\mathcal{R}_k|$ the number of derangements such that k is in the nth position.

Note that $r_1 = r_2 = ... = r_{n-1}$ (while $r_n = 0$).

Then $D_n = r_1 + ... + r_{n-1} = r_{n-1} + ... + r_{n-1} = (n-1)r_{n-1}$.

Thus we have (hopefully) simplified our problem to showing that $D_{n-2} + D_{n-1} = r_{n-1} =$ the number of derangements such that $n-1$ is in the nth position.

We need to partition the permutations in $\mathcal{R}_{n-1}$ into two sets, one with D_{n-2} elements and the other with D_{n-1} elements.

We can easily take care of D_{n-2}. The numbers $n-1$ and n do not appear in any derangement of $\{1, ..., n-2\}$. In $\mathcal{R}_{n-1}$, $n-1$ appears in the last position. We can take a look at the derangements in $\mathcal{R}_{n-1}$, such that n appears in the $(n-1)$st position. If we remove the nth and $(n-1)$st entries, we obtain a derangement in $\mathcal{D}_{n-2}$.

Ex: for $n = 5$, $23154 \in \mathcal{R}_{n-1} \to 231 \in \mathcal{D}_{n-2}$.

Thus $D_{n-2} =$ the number of derangements of $\mathcal{R}_{n-1}$ such that n is in the $(n-1)$st position (and by definition of $\mathcal{R}_{n-1}$, $n-1$ is in the nth position).

We can now look at the remaining derangements in $\mathcal{R}_{n-1}$ where n is not in the $(n-1)$st position.

Let $\mathcal{P}_n$ the set of derangement where $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \neq n, n-1$ (I.e, $k \leq n-2$).

We would like to show that $D_{n-1} = |\mathcal{P}_n| =$ the number of derangements of $\{1, ..., n\}$ such that $n-1$ is in the nth position and k is in the $(n-1)$st position for some $k \leq n-2$

Let $\mathcal{D}_{n-1} =$ the set of derangements of $\{1, ..., n-1\}$.

We would like to create a bijection from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$

Note that the differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. A derangement in $\mathcal{P}_n$ has n terms, while a derangement in $\mathcal{D}_{n-1}$ has $n-1$ terms. Thus we need to remove a term to go from $\mathcal{P}_n$ to $\mathcal{D}_{n-1}$.

If $i_1 i_2 ... i_n \in \mathcal{P}_n$, then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$. Also $i_j = n$ for some j.

In $\mathcal{D}_{n-1}$, $i_{n-1} = k$ for some $k \leq n-2$ (by definition of derangement of $\{1, ..., n-1\}$, so we have no problems with the $(n-1)$st term.

However, we have the following differences between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$:

$i_1 i_2 ... i_n$ has n terms and
n appears somewhere in $i_1 i_2 ... i_n$, and
$i_n = n-1$, so the placement of $n-1$ doesn't vary.
We can fix this by removing the nth term and replacing $i_j = n$ with $i_j = n-1$

Let $i_1 i_2 ... i_n \in \mathcal{P}_n$. Then $i_n = n-1$ and $i_{n-1} = k$ for some $k \leq n-2$.

Create $a_1a_2...a_{n-1}$, a derangement of $\{1,...,n-1\}$ by

let $a_l = \begin{cases} i_l & \text{if } i_l \neq n,\ 1 \leq l \leq n-1 \\ n-1 & \text{if } i_l = n \end{cases}$

Ex: For $n = 5$, $25314 \in \mathcal{P}_n \to 2431 \in \mathcal{D}_{n-1}$.

This gives us a bijection between $\mathcal{P}_n$ and $\mathcal{D}_{n-1}$. Thus $D_{n-1} = |\mathcal{P}_n|$.

Thus we have shown that $D_n = (n-1)r_{n-1} = (n-1)(D_{n-2} + |\mathcal{P}_n|) = (n-1)(D_{n-2} + D_{n-1})$ for $n \geq 3$.

Another (simpler) recurrance relation:

Lemma B: $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$

Proof by induction on n.

$n = 2$: $D_2 = 1$ (use definition or Thm 6.3.1)
$2D_1 + (-1)^2 = 2(0) + 1 = 1$.
Thus $D_n = nD_{n-1} + (-1)^n$ holds for $n = 2$.

Suppose $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.

By lemma A, $D_k = (k-1)D_{k-2} + (k-1)D_{k-1}$

By the induction hypothesis, $D_{k-1} = (k-1)D_{k-2} + (-1)^{k-1}$.
Thus $(k-1)D_{k-2} = D_{k-1} - (-1)^{k-1} = D_{k-1} + (-1)^k$

Thus $D_k = D_{k-1} + (-1)^k + (k-1)D_{k-1} = kD_{k-1} + (-1)^k$