6.2: Combinations with repetitions.

What is the number of $r-$combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$?

If $k = 1$; The number of $r-$combinations of $\{n \cdot a\} =$ ________

If $r = \sum_{i=1}^{k} n_i$:

The # of $r-$combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\} =$ ________

If $r > \sum_{i=1}^{k} n_i$:

The # of $r-$combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\} =$ ________

If $n_i = 1\ \forall i$:

The number of $r-$combinations of $\{1, 2, ..., k\} =$ ________

If $n_i \geq r\ \forall i$:

The number of $r-$combinations of $\{n_1 \cdot 1,\ n_2 \cdot 2, ...,\ n_k \cdot k\}$

$$= \text{the number of integral solutions to } \sum_{i=1}^{k} x_i = r$$

$$= \text{the number of permutations of } \{r \cdot 1, (k-1) \cdot +\}$$

$$= \binom{r+k-1}{r} = \frac{(r+k-1)!}{r!(k-1)!}$$

What if $2 \leq n_i < r - 1$**?**

Ex: The number of 10-combinations of $\{3 \cdot a,\ 9 \cdot b\}$

This problem is small enough to break into cases & use **sum rule:**

A 10-subset of $\{3 \cdot a,\ 9 \cdot b\}$ is of the form $\{n_1 \cdot a,\ n_2 \cdot b\}$ where

$n_1 = 0:$ $\qquad$ $n_1 = 1:$ $\qquad$ $n_1 = 2:$ $\qquad$ $n_1 = 3:$

Thus the number of 10-combinations of $\{3 \cdot a,\ 4 \cdot b\} =$ ________

Can also use inclusion-exclusion:

Let $S =$ set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$, $|S| =$ ________

Let $A_1 =$ set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$
containing more than 3 a's , $|A_1| =$ ________

I.e., A_1 can have 4, 5, 6, 7, 8, 9, or 10 a's. $\qquad$ $10 - 3 = 7.$

I.e., $|A_1| =$ the number of 6 combinations of $\{\infty \cdot a,\ \infty \cdot b\} =$ ________

Let $A_2 =$ set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$
containing more than 9 b's, $|A_2| =$ ________

$A_1 \cup A_2 =$ set of all 10-combinations of $\{\infty \cdot a,\ \infty \cdot b\}$
containing more than 3 a's and more than 9 b's $=$ ________

$\overline{A_1 \cup A_2}$ = The number of 10-combinations of $\{3 \cdot a,\ 9 \cdot b\}$

$$= \frac{(10+1)!}{10!1!} - (10-3) - (10-9) + 0 = 11 - 8 = 3$$

What if $3 \le n_i \le r-1$? Use inclusion-exclusion (Usually).

Ex: The number of integral solutions to $\sum_{i=1}^{5} x_i = 20$
where $-2 \le x_i \le 7\ \forall i$

$=$ The number of integral solutions to $\sum_{i=1}^{5} y_i = 30$
where $0 \le y_i \le 9\ \forall i$

$=$ The number of 30-combinations of the
multiset $\{9 \cdot a_1,\ 9 \cdot a_2,\ 9 \cdot a_3,\ 9 \cdot a_4,\ 9 \cdot a_5\}$

Pf: Let $y_i = x_i + 2$

See example in book for case when the number of a_i's is not the same $\forall i$.

Let $S =$ the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$
where $0 \le y_i\ \forall i$

Then $|S| =$ the number of permutations of $\{30 \cdot 1, 4 \cdot +\} =$

For $i = 1, 2, 3, 4, 5$,
let $A_i =$ the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$
where $10 \le y_i$

Ex: $(10, 5, 5, 5, 5) \in A_1$, $(0, 20, 7, 2, 1) \in A_2$,
$(0, 0, 10, 10, 10) \in A_3 \cap A_4 \cap A_5$

Then $\overline{\cup_{i=1}^{5} A_i} =$ the set of of integral solutions to $\sum_{i=1}^{5} y_i = 30$
where $0 \le y_i \le 9\ \forall i$

Note $|A_1| = |A_2| = |A_3| = |A_4| = |A_5|$.

Note $|A_1|$ = The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$ containing more than 9 a_1's.

The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$ containing at least 10 a_1's.

= The number of 20-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$

= the set of integral solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 30$ where $y_1 \geq 10$ and $y_i \geq 0$ for $i = 2, 3, 4, 5$

= the set of integral solutions to $z_1 + z_2 + z_3 + z_4 + z_5 = 20$ where $z_i \geq 0$ for $i = 1, 2, 3, 4, 5$

=

Note $|A_i \cap A_j| = |A_1 \cap A_2|$ for $i \neq j$.

The number of 30-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\}$ containing at least 10 a_1's and at least 10 a_2's

The number of 10-combinations of the multiset $\{\infty \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \infty \cdot a_4, \infty \cdot a_5\} =$

Etc. (see class notes).

Read book for examples where the pairwise intersections are not all identical