4.5.

Thm: Suppose that X is a finite totally ordered set. Then X has a maximal element $c \in X$ such that $x < c$ for all $x \in X - \{c\}$. Similarly, X has a minimal element $a \in X$ such that $a < x$ for all $x \in X - \{a\}$.

Proof by induction on $|X| = n$.

$n = 1$. If $X = \{x_1\}$, then x_1 is both the minimal and maximal of X.

$n = k$. Suppose for $|X| = k$ that X has a maximal element.

$n = k + 1$. Suppose $|X| = k + 1$.

Let $b \in X$. Then $|X - \{b\}| = k$.

Thus $X - \{b\}$ has a maximal element $c \in X - \{b\}$.

Suppose $b < c$. Then c is the maximal element of X.

Suppose $c < b$. For all $x \in X - \{b, c\}$, $x < c$.

By transitivity $x < b$.

Thus b is the maximal element of X.

Similarly, X has a minimal element $a \in X$ such that $a < x$ for all $x \in X - \{a\}$.

5.1 Patterns from Pascal's triangle

We create the table below where the entry in the nth row and kth column is

$$C(n,k) = C(n-1,k) + C(n-1,k-1).$$

$C(n,0) = 1 = \#$ of 0-element subsets of S where $|S| = n$.

$C(n,n) = 1 = \#$ of n-element subsets of S where $|S| = n$.

Table for $C(n,k)$

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

$1+2=3; \quad 1+2+3=6; \quad 1+2+3+4=10; \quad \cdots$

Observe symmetry: $\binom{n}{k} = \binom{n}{n-k}$

Sum any row: $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

5.2:

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)(x+y) = x^3 + 3x^2y + 3xy^2 + y^3$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y) \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

Thm 5.2.1: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Proof Outline:

The terms of $(x+y)^n$ are of the form $x^k y^{n-k}$.

The coefficient of $x^k y^{n-k}$

$=$ the number of ways to choose k x's and $(n-k)$ y's

$=$ the number of ways to choose k x's from n x's $= \binom{n}{k}$.

Alternatively,

The coefficient of $x^k y^{n-k}$

$=$ the number of ways to choose k x's and $(n-k)$ y's

$=$ the number of permutations of the multiset $\{k \cdot x, (n-k) \cdot y\} = \binom{n}{k}$.

2nd proof of Thm 5.2.1: Induction (read textbook).

Obtain other formulas via substitution and algebraic manipulation including differentiation.

Cor 5.2.2: $\quad (1+x)^n = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Let $x = 1: \quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$

Let $x = -1: \quad 0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

I.e., $\; 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - ... + (-1)^n \binom{n}{n}$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k$$

$\lfloor x \rfloor$ = floor of x = max $\{n \in \mathbb{Z} \mid n \leq x\}$

$\lceil x \rceil$ = ceiling of x = min $\{n \in \mathbb{Z} \mid n \geq x\}$

Thus $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k = \frac{1}{2}(2^n) = 2^{n-1}$

and $$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k = \frac{1}{2}(2^n) = 2^{n-1}$$