Thm 4.5.1: Suppose $|X| = n$. Then there exists a bijection between the total orders of X and the permutations of X. Hence there exists $n!$ different total orders on n.

Proof: Suppose $X = \{1, ..., n\}$ and suppose $f(1), f(2), ..., f(n)$ is a permutation of the elements of X.

Claim: $f(1) \leq f(2) \leq \ ... \ \leq f(n)$ defines a total order.

Note the above claim is equivalent to:

Claim: $f(i) \leq f(j)$ iff $i \leq j$ defines a total order on X.

Proof of claim:

Claim: $\leq$ is reflexive. That is, $\forall x \in X$, $x \leq x$.

Claim: $\leq$ is anti-symmetric. I.e., if $x \leq y$ and $y \leq x$, then $x = y$.

Claim: $\leq$ is transitive. That is, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Thus $\leq$ is a partial order. Note every pair of elements of X is comparable. Thus $\leq$ is a total order.

Suppose we have a total order $\leq$ on X.

Claim: We can order the elements of X as follows:

$$f(1) \leq f(2) \leq ... \leq f(n) \text{ for some permutation of } X.$$

Proof by induction on $n = |X|$.

Suppose $n = 1$:

Suppose that if $|X| = n - 1$, we can order the elements of X as follows: $f(1) < f(2) < ... < f(n-1)$ for some permutation of X.

Suppose $|X| = n$.

Note that we have shown a 1:1 correspondence between permutations of X and total orders of X. Hence there exists $n!$ different total orders on n.

Defn: An *equivalence relation* is reflexive, symmetric, transitive.

Ex: $\cong_p$ is an equivalence relation where $x \cong_p y$ if $\frac{x-y}{p} \in \mathbb{Z}$

Claim: $\cong_p$ is reflexive. That is, $\forall x \in X$, $x \cong_p x$.

Claim: $\cong_p$ is symmetric. I.e., if $x \cong_p y$, then $y \cong_p x$.

Claim: $\cong_p$ is transitive. I.e., if $x \cong_p y$ and $y \cong_p z$, then $x \cong_p z$.

Thus $\cong_p$ is an equivalence relation.

Equivalence class $[a] = \{x \mid x \sim a\}$

For $\cong_2$

$[4] =$

$[-2] =$

$[1] =$

Ex: $\mathbb{Z} =$

$\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X iff
$X = \bigcup_{P_\alpha \in \mathcal{P}} P_\alpha$, $P_\alpha \neq \emptyset \ \forall \alpha$, and $P_\alpha \cap P_\beta \neq \emptyset$ implies $P_\alpha = P_\beta$

Thm 4.5.3: If $\sim$ is an equivalence relation on X, then
$\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

If $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ is a partition of X, then
$x \sim y$ iff $\exists P_\alpha$ such that $x, y \in P_\alpha$ is an equivalence relation.

Proof: Suppose $\sim$ is an equivalence relation on X.

Claim: $\{[x_\alpha] \mid x_\alpha \in X\}$ is a partition of X.

Let $x_\alpha \in X$. Then $x_\alpha \in [x_\alpha]$ since $\sim$ is reflexive.
Thus $[x_\alpha] \neq \emptyset$ and $X = \bigcup_{x_\alpha \in X} [x_\alpha]$.

Suppose $[x_\alpha] \cap [x_\beta] \neq \emptyset$.

Claim: $[x_\alpha] = [x_\beta]$

Claim: $[x_\alpha] \subset [x_\beta]$ and $[x_\beta] \subset [x_\alpha]$

Claim: If $z \in [x_\alpha] = \{x \mid x \sim x_\alpha\}$, then $z \in [x_\beta] = \{x \mid x \sim x_\beta\}$

Proof of Claim: Since $z \in [x_\alpha]$, $z \sim x_\alpha$. Since

Thus $[x_\alpha] \subset [x_\beta]$. Similarly $[x_\beta] \subset [x_\alpha]$.

Suppose $\mathcal{P} = \{P_\alpha \mid a \in A\}$.

Claim: $x \sim y$ iff $\exists P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$ is an equivalence relation on X.

Proof of Claim: HW #44 (don't assume finite).