Defn: An *operation* on G is a map $\circ : G \times G \to G$.

Defn: $(G, \circ)$ is a *group* if

0.) G is closed under $\circ$: $f, g \in G$ implies $f \circ g \in G$.

1.) $\circ$ is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.

2.) G has an identity element: $\exists i \in G$ such that $\forall\ f \in G$,

$$i \circ f = f \circ i = f.$$

3.) All elements of G are invertible: for all $f \in G$, $\exists f^{-1} \in G$ such that, $f^{-1} \circ f = f \circ f^{-1} = i$.

Ex: $(\mathcal{Z}, +)$, $\quad (\mathcal{R} - \{\ 0\ \}, \times)$

Ex: $(\mathcal{M}, \times)$ where $\mathcal{M}$ = set of invertible matrices.

Ex: Let $G = \{f : A \to A \mid f \text{ is a bijection }\}$ under composition of functions.

Ex: S_n = the group of permutations of $\{1, ..., n\}$

$$= \{f : \{1, ..., n\} \to \{1, ..., n\} \mid f \text{ is a bijection }\}$$

Ex: The group of rotations of a regular $n-$gon

$$= \{i, \rho_n, \rho_n^2, ..., \rho_n^{n-1}\}$$

Ex: The set of reflections of a regular $n-$gon is NOT a group (the product of two reflections is a rotation).

Ex: D_n = the group of symmetries of a regular $n-$gon

= the group of rotations and reflections of a regular $n-$gon.

$$= \{i,\ \rho_n,\ \rho_n^2, ...,\ \rho_n^{n-1},\ \tau,\ \rho_n\tau,\ \rho_n^2\tau, ...,\ \rho_n^{n-1}\tau\}$$

Note a group need not be commutative: $f \circ g \neq g \circ f$.

Defn: H is a *subgroup* of G if $H \subset G$ and H is a group. I.e.,

0.) H is closed under $\circ$.
1.) the identity i of G is in H.
2.) for all $f \in H$, $f^{-1} \in H$.

Ex: D_n is a subgroup of S_n.
Note: $D_3 = S_3$. For $n > 3$ $D_n \subset S_n$, $D_n \neq S_n$.

Defn: Let X be a set and G a group. An *action of G on X* is a map $* : G \times X \to X$ such that

1.) $e * x = x \quad \forall x \in X$.

2.) $(g \circ f) * x = g * (f * x) \quad \forall x \in X$ and $\forall g, f \in G$.

Let C be a set of colors.
Defn: A *coloring* of X is a function $\mathbf{c} : X \to C$

Example: If $X = \{1, 2, 3\}$, $C = \{\text{red, blue}\}$, then let $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}, \mathbf{c_5}, \mathbf{c_6}, \mathbf{c_7}, \mathbf{c_8}\}$ where

$$\mathbf{c_i} : \{1, 2, 3\} \to \{\text{red, blue}\} \ \forall i \text{ and}$$

$\mathbf{c_1}(j) =$ blue for all j;

$\mathbf{c_2}(1) =$ blue, $\mathbf{c_2}(2) =$ blue , $\mathbf{c_2}(3) =$ red;

$\mathbf{c_3}(1) =$ blue, $\mathbf{c_3}(2) =$ red, $\mathbf{c_3}(3) =$ blue;

$\mathbf{c_4}(1) =$ red, $\mathbf{c_4}(2) =$ blue, $\mathbf{c_4}(3) =$ blue;

$\mathbf{c_5}(1) =$ blue, $\mathbf{c_5}(2) =$ red, $\mathbf{c_5}(3) =$ red;

$\mathbf{c_6}(1) =$ red, $\mathbf{c_6}(2) =$ blue, $\mathbf{c_6}(3) =$ red;

$\mathbf{c_7}(1) =$ red, $\mathbf{c_7}(2) =$ red, $\mathbf{c_7}(3) =$ blue;

$\mathbf{c_8}(j) =$ red for all j.

Let G be a set of permutations.
A permutation f acts on a coloring $\mathbf{c}$ as follows:

$$(f * \mathbf{c})(x) = (\mathbf{c} \circ f^{-1})(x) = \mathbf{c}(f^{-1}(x))$$

Note: $id * \mathbf{c} = \mathbf{c} \circ id^{-1} = \mathbf{c} \circ id = \mathbf{c}$
Also, $(g \circ f) * \mathbf{c} = \mathbf{c} \circ (g \circ f)^{-1} = \mathbf{c} \circ (f^{-1} \circ g^{-1})$

$$= (\mathbf{c} \circ f^{-1}) \circ g^{-1} = (f * \mathbf{c}) \circ g^{-1} = g * (f * \mathbf{c})$$

Ex: Suppose f is the permutation $\rho_3 = 231$. Then

$$\rho_3 * \mathbf{c_2}(1) = \mathbf{c_2}(\rho_3^{-1}(1)) = \mathbf{c_2}(3) = \text{red.}$$
$$\rho_3 * \mathbf{c_2}(2) = \mathbf{c_2}(\rho_3^{-1}(2)) = \mathbf{c_2}(1) = \text{blue.}$$
$$\rho_3 * \mathbf{c_2}(3) = \mathbf{c_2}(\rho_3^{-1}(3)) = \mathbf{c_2}(2) = \text{blue.}$$

Thus $\rho_3 * \mathbf{c_2} = \mathbf{c_4}$

Defn: Let G be a subgroup of the set of permutations, S_n.
$\mathbf{c_1} \sim \mathbf{c_2}$ if there exists an $f \in G$ such that $f * \mathbf{c_1} = \mathbf{c_2}$

Theorem: $\sim$ is an equivalence relation.

Ex: Find the number of circular permutations of the multiset $\{2 \cdot blue, 1 \cdot red\}$

14.2: Burnside's Theorem.

Defn: The *stabilizer* of $\mathbf{c} = G(\mathbf{c}) = \{f \in G \mid f * \mathbf{c} = \mathbf{c}\}$.

Defn: $\mathcal{C}(f) = \{\mathbf{c} \in \mathcal{C} \mid f * \mathbf{c} = \mathbf{c}\}$.

Thm 14.2.1a: $G(\mathbf{c})$ is a group.

Thm 14.2.1b: $g * \mathbf{c} = f * \mathbf{c}$ if and only if $f^{-1} \circ g \in G(\mathbf{c})$.

Thm 14.2.2: $|\{f * \mathbf{c} \mid f \in G\}| = \frac{|G|}{|G(\mathbf{c})|}$

Note $[\mathbf{c}] = |\{f * \mathbf{c} \mid f \in G\}|$

= the # of different colorings which are equivalent to $\mathbf{c}$.

= the number of elements in the equivalence class $[\mathbf{c}]$.

Thm 14.2.3: Suppose for all $f \in G$ and for all $\mathbf{c} \in \mathcal{C}$, $f * \mathbf{c} \in \mathcal{C}'$. Then

$N(G, C)$ = the number of non-equivalent colorings in $\mathcal{C}$

= the number of different equivalence classes

$$= \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)|$$

= the average of the # of colorings fixed by the permutations in G.