$C^\infty(M) = \{g \mid g^{smooth} : M \to \mathbf{R}\}$

D is a derivation iff $D : C^\infty(p) \to \mathbf{R}$ and D is linear and satisfies the Leibniz rule.

That is D is a derivation if $D(f) \in \mathbf{R}$,
$D(cf) = cD(f)$, $D(f+g) = D(f) + D(g)$,
$D(fg) = f(p)Dg + g(p)Df$

Defn: A *vector field* or *section of the tangent bundle* TM is a smooth function
s: $M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

Ex: If $M = \mathbf{R}$, let $s(p) = (p, (\frac{d}{dx})_p)$

Sometimes we will drop the p and write $s(p) = (\frac{d}{dx})_p$

Let $f \in C^\infty(\mathbf{R})$. For all $p \in \mathbf{R}$, $s(p)(f) = (\frac{df}{dx})_p = \frac{df}{dx}(p)$

Define $s_f : \mathbf{R} \to \mathbf{R}$, $s_f(p) = \frac{df}{dx}(p)$. I.e., $s_f = \frac{df}{dx}$

Note s_f is smooth.

Lemma 3.4.1: For any vector field s and smooth functions f and g on M, we have

$$s_{fg}(p) = f(p) \cdot s_g(p) + s_f(p) \cdot g(p)$$

Proof: $\frac{d(fg)}{dx}(p) = f(p)\frac{dg}{dx}(p) + \frac{df}{dx}(p)g(p)$

We can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Ex: $S : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$, $S(f) = \frac{df}{dx}$. I.e., $S = \frac{d}{dx}$

Ex: If $M = \mathbf{R}$, then $s(p) = a(p)(\frac{d}{dx})_p$ where $a : \mathbf{R} \to \mathbf{R}$ is a smooth function.

Let $f \in C^\infty(\mathbf{R})$.
For all $p \in \mathbf{R}$, $s(p)(f) = a(p)(\frac{df}{dx})_p = a(p)\frac{df}{dx}(p)$

Define $s_f : \mathbf{R} \to \mathbf{R}$, $s_f(p) = a(p)\frac{df}{dx}(p)$. I.e., $s_f = a\frac{df}{dx}$

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Lemma 3.4.1: For any vector field s and smooth functions f and g on M, we have

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We can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Ex: $S : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$, $S(f) = a\frac{df}{dx}$ I.e., $S = a\frac{d}{dx}$

In the above we used the charts $\phi_p : \mathbf{R} \to \mathbf{R}, \phi_p(x) = x - p$.

Thus $\frac{d(g(\phi_p^{-1}(x)))}{dx}|_{x=0} = \frac{d(g(x+p))}{dx}|_{x=0} = \frac{dg}{dx}(p)$

Note $\phi_0(x) = \phi_p(x+p)$.

Thus $\frac{d(\phi_p(\phi_0^{-1}(x))}{dx}|_{x=0} = \frac{d(\phi_p(\phi_p^{-1}(x+p)))}{dx}|_{x=0} = \frac{d(x+p)}{dx}|_{x=0} = 1$

If we use the chart $\psi_q : \mathbf{R} \to \mathbf{R}, \psi_q(x) = q - x$.

Then $\frac{d(g(\psi_p^{-1}(x)))}{dx}|_{x=0} = \frac{d(g(p-x))}{dx}|_{x=0} = \frac{-dg}{dx}(p)$

Note $\frac{d(\psi_q(x+p))}{dx}|_{x=0} = \frac{d\psi_q}{dx}|_p = \frac{d(q-x)}{dx}|_p = -1$

Example of a non-smooth vector field on $\mathbf{R}$:

If $p \geq 0$, let $s(p) = (p, (\frac{d}{dx})_p)$
[i.e., the basis element of $T_p(\mathbf{R})$ from ϕ_p]

If $p < 0$, let $s(p) = (p, (-\frac{d}{dx})_p)$
[i.e., the basis element of $T_p(\mathbf{R})$ from ψ_p]

Ex: If $M = \mathbf{R}^2$, then $s(\mathbf{p}) = a(\mathbf{p})(\frac{\partial}{\partial x})_{\mathbf{p}} + b(\mathbf{p})(\frac{\partial}{\partial y})_{\mathbf{p}}$ where $a, b : \mathbf{R}^2 \to \mathbf{R}$ are smooth functions.

Ex: Let $\{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_m})_p\}$ be a basis for $T_p(M)$.

Let $s : M \to TM$, $s(p) = (p, \Sigma_{i=1}^m a_i(p)(\frac{\partial}{\partial x_i})_p)$