$G(p,N) = \{[g] \mid g^{smooth} : U \to N$, for some U^{open} such that $p \in U \subset M\}$

$G(p) = G(p, \mathbf{R})$

$C^\infty(M) = \{g \mid g^{smooth} : M \to \mathbf{R}\}$

$C^\infty(p) = \{g \mid g^{smooth} : U \to N$, for some U^{open} such that $p \in U \subset M\}$

D is a derivation iff $D : C^\infty(p) \to \mathbf{R}$ and D is linear and satisfies the Leibniz rule.

That is D is a derivation if $D(f) \in \mathbf{R}$,
$D(cf) = cD(f)$, $D(f+g) = D(f) + D(g)$,
$D(fg) = f(p)Dg + g(p)Df$

Let $\alpha : I \to M$ where $I =$ an interval $\subset \mathbf{R}$, $\alpha(0) = p$. Note $[\alpha] \in G[0, M]$

Directional derivative of $[g]$ in direction $[\alpha] =$

$$D_\alpha g = \frac{d(g \circ \alpha)}{dt}|_{t=0} \in \mathbf{R}$$

$D_\alpha : G(p) \to \mathbf{R}$ is a derivation.

$T_p(M) = \{v : G(p) \to \mathbf{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

$v \in T_p(M)$ is called a *derivation*

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$,

the *standard basis* for $T_p(M) = \{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_m})_p\}$, where $(\frac{\partial}{\partial x_i})_p = D_{\alpha_i}$

and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

If $v \in T_p(M)$, then $v = \Sigma_{i=1}^m a_i (\frac{\partial}{\partial x_1})_p$ where $a_i = v([\pi_i \circ \phi])$

$TM = \cup_{p \in M} T_p(M) = \{(p, v) \mid p \in M, v \in T_pM\}$,

let π: $TM \to M$ be defined by $\pi(p, v) = p$.

Let (ϕ, U) be a chart for M.

If $q \in U$, let $\{(\frac{\partial}{\partial x_1})_q, ..., (\frac{\partial}{\partial x_m})_q\}$ be the standard basis (w.r.t (ϕ, U)) for $T_q(M) = T_q$

t_ϕ: $\pi^{-1}(U) \to \phi(U) \times \mathbf{R}^m \subset \mathbf{R}^{2m}$,

$t_\phi(q, v) = (\phi(q), a_1, ..., a_m)$ where $v = \Sigma_{i=1}^m a_i (\frac{\partial}{\partial x_i})_q$

Let $\mathcal{A}$ be a maximal atlas for M.

Basis for topology on TM :
$\{W \mid \exists (\phi, U) \in \mathcal{A}$ s.t. $W \subset \pi^{-1}(U)$ and $t_\phi(W)$ open in $\mathbf{R}^{2m}\}$

Claim: TM is a $2m-$manifold and
$\mathcal{C} = \{(t_\phi, \pi^{-1}(U)) \mid (\phi, U) \in \mathcal{A}\}$ is a pre-atlas for TM.

π: $TM \to M$, $\pi(p, v) = p$ is smooth

$df : TM \to TN$ defined by $df(p, v) = (f(p), d_pf(v))$ is smooth if $f : M \to N$ is smooth.

Proof: See Hitchin 4.1 (in Chapter 1 of
http://www2.maths.ox.ac.uk/ hitchin/hitchinnotes/hitchinnotes.html

Defn: A *vector field* or *section of the tangent bundle TM* is a smooth function s: $M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

Ex: If $M = \mathbf{R}$, let $s(p) = (p, (\frac{d}{dx})_p)$

Sometimes we will drop the p and write $s(p) = (\frac{d}{dx})_p$

Let $f \in C^\infty(\mathbf{R})$. For all $p \in \mathbf{R}$, $s(p)(f) = (\frac{df}{dx})_p = \frac{df}{dx}(p)$

Define $s_f : \mathbf{R} \to \mathbf{R}$, $s_f(p) = \frac{df}{dx}(p)$. I.e., $s_f = \frac{df}{dx}$

Note s_f is smooth.

We can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Ex: $S : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$, $S(f) = \frac{df}{dx}$. I.e., $S = \frac{d}{dx}$

Ex: If $M = \mathbf{R}$, then $s(p) = a(p)(\frac{d}{dx})_p$ where $a : \mathbf{R} \to \mathbf{R}$ is a smooth function.

Let $f \in C^\infty(\mathbf{R})$.
For all $p \in \mathbf{R}$, $s(p)(f) = a(p)(\frac{df}{dx})_p = a(p)\frac{df}{dx}(p)$

Define $s_f : \mathbf{R} \to \mathbf{R}$, $s_f(p) = a(p)\frac{df}{dx}(p)$. I.e., $s_f = a\frac{df}{dx}$

Note s_f is smooth.

We can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Ex: $S : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R})$, $S(f) = a\frac{df}{dx}$ I.e., $S = a\frac{d}{dx}$

In the above we used the charts $\phi_p : \mathbf{R} \to \mathbf{R}, \phi_p(x) = x - p$.

Thus $\frac{d(g(\phi_p^{-1}(x)))}{dx}|_{x=0} = \frac{d(g(x+p))}{dx}|_{x=0} = \frac{dg}{dx}(p)$

Note $\phi_0(x) = \phi_p(x+p)$.

Thus $\frac{d(\phi_p(\phi_0^{-1}(x))}{dx}|_{x=0} = \frac{d(\phi_p(\phi_p^{-1}(x+p)))}{dx}|_{x=0} = \frac{d(x+p)}{dx}|_{x=0} = 1$

If we use the chart $\psi_q : \mathbf{R} \to \mathbf{R}, \psi_q(x) = q - x$.

Then $\frac{d(g(\psi_p^{-1}(x)))}{dx}|_{x=0} = \frac{d(g(p-x))}{dx}|_{x=0} = \frac{-dg}{dx}(p)$

Note $\frac{d(\psi_q(x+p))}{dx}|_{x=0} = \frac{d\psi_q}{dx}|_p = \frac{d(q-x)}{dx}|_p = -1$

Example of a non-smooth vector field on $\mathbf{R}$:

If $p \geq 0$, let $s(p) = (p, (\frac{d}{dx})_p)$
[i.e., the basis element of $T_p(\mathbf{R})$ from ϕ_p]

If $p < 0$, let $s(p) = (p, (-\frac{d}{dx})_p)$
[i.e., the basis element of $T_p(\mathbf{R})$ from ψ_p]

Ex: If $M = \mathbf{R}^2$, then $s(\psi) = a(\psi)(\frac{\partial}{\partial x})_\psi + b(\psi)(\frac{\partial}{\partial y})_\psi$ where $a, b : \mathbf{R}^2 \to \mathbf{R}$ are smooth functions.

Ex: Let $\{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_m})_p\}$ be a basis for $T_p(M)$.

Let $s : M \to TM$, $s(p) = (p, \Sigma_{i=1}^m a_i(p)(\frac{\partial}{\partial x_i})_p)$

Defn: s is *never zero* if $s(p) \neq (p, \mathbf{0})$ for all $p \in M$.

Prop: Let G be a Lie group. Then G admits a never-zero vector field.

Note: S^n admits a never-zero vector field iff n odd.

Let $p_2(s(p)) = p_2(p, v_p) = v_p$

Defn: The vector fields $s_1, ..., s_k$ are *linearly independent* iff for all $p \in M$, $p_2(s_1(p)), ..., p_2(s_k(p))$ are linearly independent.

Defn: M is parallelizable (or equivalently the "tangent bundle π: $TM \to M$ is trivial") iff TM admits m linearly independent vector fields.

Suppose M is parallelizable. Thus for each $p \in M$, let $\{v_{1,p}, ..., v_{m,p}\}$ be ANY basis for $T_p(M)$ such that $s_i : M \to TM$, $s_i(p) = (p, v_{i,p})$ is a SMOOTH vector field.

NOTE: We can form m vector fields using basis elements iff M is parallelizable.

When M is parallelizable, we can define:

$t : TM \to M \times R^m$, $t(p, v) = (p, a_1, ..., a_m)$ where $v = \Sigma_{i=1}^m a_i v_{i,p}$

Let $\rho_1 : M \times R^m \to M$, $\rho_1(p, \mathbf{x}) = p$.

$\rho_2 : M \times R^m \to R^m$, $\rho_1(p, \mathbf{x}) = \mathbf{x}$.

$\rho_1 \circ t : TM \to M$, $(\rho_1 \circ t)(p, v) = \pi(p, v) = p$

Recall $\pi^{-1}(p) = T_p(M)$

Prop: $t|_{T_p(M)} : T_p(M) \to \{p\} \times R^m$ is a linear isomorphism for all p.

or equivalently,

$\rho_2 \circ t|_{T_p(M)} : T_p(M) \to R^m$ is a linear isomorphism for all p.

since $\rho_2 \circ t|_{T_p(M)}(\Sigma_{i=1}^m a_i v_{i,p}) = (a_1, ..., a_m)$

HENCE: $t : TM \to M \times R^m$ is a diffeomorhism.

Note that for all $p \in M$, given a basis $\{v_{1,p}, ..., v_{m,p}\}$ be a basis for $T_p(M)$, we can always define a linear isomorphism:

$$t_p : T_p(M) \to R^m, \ T(\Sigma_{i=1}^m a_i v_{i,p}) = (a_1, ..., a_m)$$

However, $t : TM \to M \times R^m$, $t(p, v) = (p, t_p(v))$ may not be smooth (recall example of non-smooth vector field on p. 4).

In general TM may not be diffeomorphic to $M \times R^m$.

Randell 3.4 The bracket of two vector fields.

Defn: A *vector field* or *section of the tangent bundle* TM is a smooth function s: $M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

I.e, s takes $p \in M$ to the derivation $v_p : C^\infty(M) \to \mathbf{R}$

Let $f \in C^\infty(M)$

Define $s_f : M \to \mathbf{R}$, $s_f(p) = v_p([f])$ where $s(p) = (p, v_p)$

Note s_f is smooth.

Thus we can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Lemma 3.4.2: Let $S : C^\infty(M) \to C^\infty(M)$ be linear, and suppose $S(fg)(p) = f(p) \cdot S(g)(p) + S(f)(p) \cdot g(p)$. Then S is a vector field.

Proof: Define $s : M \to TM$, $s(p) = (p, S_p)$ where

Define $S_p : C^\infty(M) \to \mathbf{R}$, $S_p(f) = S(f)(p)$, i.e, the function $S(f)$ evaluated at p.

Claim S_p is a derivation.

Show S_p is linear and satisfies the Leibniz rule.

Claim s is smooth.

Defn: If A, B are vector fields, let $AB = A \circ B$

Defn: The *Lie Bracket* of vector fields A and B is $[A, B] = AB - BA : C^\infty(M) \to C^\infty(M)$.

Thm: The Lie bracket of vector fields is a vector field.

Let $\alpha : I \to M$ where $I =$ an interval $\subset \mathbf{R}$, $\alpha(0) = p$. Note $[\alpha] \in G[0, M]$

Directional derivative of $[g]$ in direction $[\alpha] = D_\alpha g = \frac{d(g\circ\alpha)}{dt}|_{t=0} \in \mathbf{R}$

$D_\alpha : G(p) \to \mathbf{R}$ is a derivation.

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$,

the *standard basis* for $T_p(M) = \{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_m})_p\}$, where $(\frac{\partial}{\partial x_i})_p = D_{\alpha_i}$

and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

If $v \in T_p(M)$, then $v = \Sigma_{i=1}^m a_i (\frac{\partial}{\partial x_1})_p$ where $a_i = v([\pi_i \circ \phi])$

Let (U, ϕ) be a chart for M such that $\mathbf{0} \in \phi(U)$.

Suppose $q \in U$. Choose $\epsilon > 0$ such that $B(\phi(q), \epsilon) \subset \phi(U)$ and $B(\mathbf{0}, \epsilon) \subset \phi(U)$.

Let $\tau_q : B(\phi(q), \epsilon) \to B(\mathbf{0}, \epsilon)$, $\tau_q(\mathbf{x}) = \mathbf{x} - \phi(q)$.

the *standard basis* for $T_q(M)$ with respect to $(U, \phi) =$
the standard basis for $T_q(M)$ with respect to $(\phi^{-1}(B(\mathbf{0}, \epsilon)), \tau_q \circ \phi)$

Hence the standard basis (w.r.t. (U, ϕ)) $= \{(\frac{\partial}{\partial x_1})_q, ..., (\frac{\partial}{\partial x_m})_q\}$, where $(\frac{\partial}{\partial x_i})_q = D_{\alpha_i}$

$\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(\tau^{-1}(0, ..., t, ..., 0))$

$= \phi^{-1}(\phi_1(q), ..., \phi_{i-1}(q), \phi_i(q) + t, \phi_{i+1}(q), ..., \phi_m(q))$ where $\phi_i = \pi_i \circ \phi$.

Suppose $f : M \to \mathbf{R}$ is smooth. Recall f is smooth iff for all $p \in M$, there exists a chart (U, ϕ) such that $p \in U$ and $f \circ \phi^{-1} : \phi(U) \subset \mathbf{R}^m \to \mathbf{R}$ is smooth.

Claim: $\frac{\partial f}{\partial x_i} : U \to \mathbf{R}$, $\frac{\partial f}{\partial x_i}(q) = (\frac{\partial}{\partial x_i})_q(f)$ is smooth.

NOTE: $\frac{\partial f}{\partial x_i} : U \to \mathbf{R}$ is only defined on U, and is NOT a globally defined function on M.

We will show that $\frac{\partial f}{\partial x_i} \circ \phi^{-1} : \phi(U) \subset \mathbf{R}^m \to \mathbf{R}$ is smooth. Let $q = \phi^{-1}(\mathbf{x})$

$\frac{\partial f}{\partial x_i} \circ \phi^{-1}(\mathbf{x}) = \frac{\partial f}{\partial x_i}(q) = (\frac{\partial}{\partial x_i})_q(f) = D_{\alpha_i}(f) = \frac{d(f\circ\alpha_i)}{dt}|_{t=0} =$

$\frac{d((f\circ\phi^{-1})(\phi_1(q),...,\phi_{i-1}(q),\phi_i(q)+t,\phi_{i+1}(q),...,\phi_m(q)))}{dt}|_{t=0} = \frac{\partial(f\circ\phi^{-1})}{\partial x_i}|_{\phi(q)} = \frac{\partial(f\circ\phi^{-1})}{\partial x_i}|_x$

Thus $\frac{\partial f}{\partial x_i} \circ \phi^{-1}$ is smooth since $f \circ \phi^{-1}$ is smooth.