

B is bilinear if

$$B(cv_1 + dv_2, w) = cB(v_1, w) + dB(v_2, w)$$

$$B(v, cw_1 + dw_2) = cB(v, w_1) + dB(v, w_2)$$

Thus

$$B((v_1, w_1) + (v_2, w_2)) = B(v_1 + v_2, w_1 + w_2)$$

$$= B(v_1, w_1 + w_2) + B(v_2, w_1 + w_2)$$

$$= B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2)$$

B is linear if

$$B((v_1, w_1) + (v_2, w_2)) = B((v_1, w_1)) + B((v_2, w_2))$$

$$B(c(v_1, w_1)) = cB((v_1, w_1))$$

Let $(V \times W)^b = \{B : V \times W \rightarrow \mathbf{R} \mid B \text{ bilinear} \}$

$V \otimes W = [(V \times W)^b]^* = \{h : (V \times W)^b \rightarrow \mathbf{R} \mid h \text{ linear} \}$

If $v \in V, w \in W$, let $v \otimes w : (V \times W)^b \rightarrow \mathbf{R}$, $(v \otimes w)(B) = B(v, w)$

Prop: $v \otimes w \in V \otimes W$.

Proof: $(v \otimes w)(c_1 B_1 + c_2 B_2) = (c_1 B_1 + c_2 B_2)(v, w) = c_1 B_1(v, w) + c_2 B_2(v, w) = c_1(v \otimes w)(B_1) + c_2(v \otimes w)(B_2)$.

Prop: $\sum_{i=1}^n r_i(v_i \otimes w_i) \in V \otimes W$.

Proof: $(\sum_{i=1}^n r_i(v_i \otimes w_i))(c_1 B_1 + c_2 B_2) = \sum_{i=1}^n r_i(v_i \otimes w_i)(c_1 B_1 + c_2 B_2) = \sum_{i=1}^n c_1 r_i(v_i \otimes w_i)(B_1) + c_2 r_i(v_i \otimes w_i)(B_2) = c_1(\sum_{i=1}^n r_i(v_i \otimes w_i))(B_1) + c_2(\sum_{i=1}^n r_i(v_i \otimes w_i))(B_2)$.

Note $(\sum_{i=1}^n r_i(v_i \otimes w_i))(B) = \sum_{i=1}^n r_i(v_i \otimes w_i)(B) = \sum_{i=1}^n r_i B(v_i, w_i)$

Prop: The product \otimes is bilinear:

Proof: $((c_1 v_1 + c_2 v_2) \otimes w)(B) = B(c_1 v_1 + c_2 v_2, w) = c_1 B(v_1, w) + c_2 B(v_2, w)$
and ...

Claim: $V \otimes W = \langle v \otimes w \mid v \in V, w \in W,$

$$(c_1 v_1 + c_2 v_2) \otimes w = c_1(v_1 \otimes w) + c_2(v_2 \otimes w), \\ v \otimes (c_1 w_1 + c_2 w_2) = c_1(v \otimes w_1) + c_2(v \otimes w_2) \rangle$$

$= \langle v \otimes w \mid v \in V, w \in W, (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, (cv) \otimes w = c(v \otimes w) = v \otimes (cw) \rangle$$

Let v_1, \dots, v_m be a basis for V , w_1, \dots, w_n basis for W . If B is a bilinear form, then B is uniquely determined by the nm values $B(v_i, w_j)$.

Thus $\dim(V \times W)^b = nm = \dim V \otimes W$.

Hence a basis for $V \otimes W$ is $\{v_i \otimes w_j \mid i = 1, \dots, m, j = 1, \dots, n\}$

$$\text{Thus } V \otimes W = \{ \sum c_{ij} v_i \otimes w_j \mid \begin{array}{l} (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \\ v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, (cv) \otimes w = c(v \otimes w) = v \otimes (cw) \end{array} \}$$

The universal mapping property: There exists $\phi, V \otimes W$ such that $\phi : V \times W \rightarrow V \otimes W$ is bilinear and given any bilinear map: $B : V \times W \rightarrow U$, there exists a unique linear map $\beta : V \otimes W \rightarrow U$ such that $\beta \circ \phi = B$. Moreover ϕ and $V \otimes W$ are unique in the sense that if X and ψ satisfy the universal mapping property, then there exists an isomorphism $f : V \otimes W \rightarrow X$ such that $f \circ \phi = \psi$

Let $B : V \times W \rightarrow U$.

We will define $\beta : V \otimes W \rightarrow (U^*)^* = U$ by defining $\alpha : U^* \rightarrow (V \times W)^b$ and letting $\beta = \text{dual of } \alpha$.

Take $f : U \rightarrow \mathbf{R} \in U^*$

$$\text{Then } f \circ B : V \times W \rightarrow \mathbf{R} \text{ is bilinear: } (f \circ B)(c_1 v_1 + c_2 v_2, w) = f(c_1 B(v_1, w) + c_2 B(v_2, w)) = c_1 (f \circ B)(v_1, w) + c_2 (f \circ B)(v_2, w)$$

$$\text{Similarly } (f \circ B)(v, c_1 w_1 + c_2 w_2) = c_1 (f \circ B)(v, w_1) + c_2 (f \circ B)(v, w_2)$$

Thus $f \circ B \in (V \times W)^b$

$$\alpha : U^* \rightarrow (V \times W)^b, \alpha(f) = f \circ B$$

Let $\beta : V \otimes W \rightarrow (U^*)^*, \beta = \text{dual of } \alpha$.

Recall that if $\alpha : U^* \rightarrow (V \times W)^b$ is a linear map between vector spaces, the dual map is $\beta : V \otimes W \rightarrow (U^*)^*, \beta(g) = g \circ \alpha$.

Recall $(U^*)^*$ is naturally isomorphic to U via $d : U \rightarrow (U^*)^*, d(v) = h$ where $h : U^* \rightarrow R, h(f) = f(v)$.

$$\beta(\phi(v, w)) = \beta(v \otimes w) = (v \otimes w) \circ \alpha$$

$$[(v \otimes w) \circ \alpha](f) = (v \otimes w)(f \circ B) = f(B(v, w))$$

Note $d : U \rightarrow (U^*)^*, d(B(v, w)) = h$ where $h : U^* \rightarrow R, h(f) = f(B(v, w))$.

M is a p -fold multi-linear form on V , if $M : V^p \rightarrow \mathbf{R}$, and

$$M(v_1, \dots, v_{i-1}, c_1 v_i + c_2 w_i, v_{i+1}, \dots, v_p) = \\ c_1 M(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_p) + c_2 M(v_1, \dots, v_{i-1}, w_i, v_{i+1}, \dots, v_p)$$

Recall $(V \times W)^b = \{B : V \times W \rightarrow \mathbf{R} \mid B \text{ bilinear} \}$

$$V \otimes W = [(V \times W)^b]^* = \{h : (V \times W)^b \rightarrow \mathbf{R} \mid h \text{ linear} \}$$

Let $(V^n)^{\{m\}} = \{M : V^n \rightarrow \mathbf{R} \mid M \text{ multilinear} \}$

$$(\otimes^p V) = [(V^n)^{\{m\}}]^* = \{h : (V^n)^{\{m\}} \rightarrow \mathbf{R} \mid h \text{ linear} \}$$

M is an alternating multilinear form on V if M is multilinear and $M(v_1, \dots, v_p) = 0$ if $v_i = v_j$ for some $i \neq j$.

The tensor algebra = $T(V) = \bigoplus_{k=0}^{\infty} (\otimes^k V)$

$$c_0 + c_1 v_i + \sum c_{ij} v_i \otimes v_j + \dots + \sum c_{i_1 \dots i_p} v_{i_1} \otimes \dots \otimes v_{i_p}$$

$$(v_1 \otimes \dots \otimes v_p)(u_1 \otimes \dots \otimes u_q) = v_1 \otimes \dots \otimes v_p \otimes u_1 \otimes \dots \otimes u_q$$

A subring I of a ring R is an *ideal* of R if $ar \in I$ and $ra \in I$ for all $a \in I, r \in R$.

Let $I(V) = \text{ideal generated by } \{v \otimes v \mid v \in V\}$

The exterior algebra of V is the quotient $\Lambda^* V = T(V)/I(V)$

Let $\pi : T(V) \rightarrow \Lambda^* V$ be the quotient map.

The p -fold exterior power of V is $\Lambda^p V = \pi(\otimes^p V)$

The exterior product of $\alpha = \pi(a) \in \Lambda^p V$ and $\beta = \pi(b) \in \Lambda^q V$ is

$$\alpha \wedge \beta = \pi(a \otimes b)$$