Thm. $f : M \to N$ embedding implies $f(M)$ is a submanifold of N.

Recall K is a submanifold of N if $\forall q \in K \subset N$, $\exists g^{smooth} : V^{open} \subset N \to \mathbf{R}^{n-m}$, $q \in V$ such that $K \cap V = g^{-1}(0)$ and rank $d_p g = n - m$

Proof. Since $f : M \to N$ embedding, $f : M \to N$ is a 1-1 immersion and

$f : M \to f(M)$ is a homeomorphism where $f(M)$ is a subspace of N

Take $q \in f(M)$.

Since f is 1:1, $\exists! p \in M$ such that $f(p) = q$.

$f : M \to N$ an immersion implies f has rank $m \leq n$.

Thus by the rank theorem,

Defn. Suppose $f : M \to N$ is smooth.

$p \in M$ is a *critical point* and $f(p)$ is a *critical value* if rank $df_p < n$.

If $p \in M$ is not a critical point, then it is a *regular point.*

If $q \in N$ is not a *critical value*, then it is a *regular value.*

Note: $q \in N$ is a regular value iff $f^{-1}(q) = \emptyset$ or $\forall p \in f^{-1}(q)$, $df_p = n$.

Thm 2.3.13: Let q be a regular value of f: $M \to N$. Then either $f^{-1}(q) = \emptyset$ or $f^{-1}(q)$ is an $(m - n)$-submanifold of M.

$Gl(n, \mathbf{R})$ is an n^2 manifold.

$A \in Gl(n, \mathbf{R})$ is orthogonal if $A^t A = I$.

The orthogonal group =
$O(n) = \{A \in Gl(n, \mathbf{R}) \mid A^t A = I\}$

The special orthogonal group =
$SO(n) = \{A \in O(n) \mid det(A) = 1\}$

$O(n)$, $SO(n)$ are subgroups of $Gl(n, \mathbf{R})$.

$O(n)$, $SO(n)$ are closed in $Gl(n, \mathbf{R})$.

If $A \in O(n)$, then $det(A) = \pm 1$

$SO(n)$ is open in $0(n)$.

$s : Gl(n, \mathbf{R}) \to Gl(n, \mathbf{R})$, $s(A) = A^t A$ is smooth.

Let $\mathcal{S} =$ the set of symmetric matrices.

Then $\mathcal{S} =$ is an manifold.

$s : Gl(n, \mathbf{R}) \to \mathcal{S}$, $s(A) = A^t A$ is smooth.

$s^{-1}(I) =$

Claim: I is a regular value of $s : Gl(n, \mathbf{R}) \to \mathcal{S}$, $s(A) = A^t A$.

That is, if $A \in O(n)$, $d_A S$ has rank $\frac{n(n+1)}{2}$.

$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Thus if I is a regular value, $O(n)$ is an $\frac{n(n-1)}{2}$ submanifold of $Gl(n, \mathbf{R})$.