Randell 3.3

Defn: A *flow* on M is a smooth action of the Lie group $\mathbf{R}^1$ on M, σ: $\mathbf{R}^1 \times M \to M$.

A flow is also called a *dynamical system.*

$\sigma(t, m) = \sigma_t(m)$

$\sigma_0(m) = m, \qquad \sigma_t \circ \sigma_s(m) = \sigma_{t+s}(m) = \sigma_s \circ \sigma_t(m)$

$\sigma_{-t} = \sigma_t^{-1}$

$\sigma_t : M \to M$ is a diffeomorphism.

Ex: $\sigma : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$, $\sigma_t(x, y) = (x, y) + t(1, 2)$

Defn: The *orbit* of $x \in M =$
$\mathbf{R}(x) = \{y \in M \mid \exists t \in \mathbf{R} \text{ such that } y = tx\}$

A *flow line* is the smooth path $\alpha_p : \mathbf{R} \to M$, $\alpha_p(t) = \sigma(t, p)$.

Prop: each $q \in M$ lies on a unique flow line.

“differentiating along the flow”: Given a flow σ on M, define $s_\sigma \colon M \to TM$ by $s_\sigma(p) = (p, d\alpha_p/dt|_{\,t=0})$

Proposition 3.3.3: s_σ is a section of TM.

Randell 3.4 The bracket of two vector fields.

$C^\infty(M) = \{g \mid g^{smooth} : M \to \mathbf{R}\}$

Defn: A *vector field* or *section of the tangent bundle* TM is a smooth function
s: $M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

I.e, s takes $p \in M$ to the derivation $v_p : C^\infty(M) \to \mathbf{R}$

Let $f \in C^\infty(M)$

Define $s_f : M \to \mathbf{R}$, $s_f(p) = v_p([f])$ where $s(p) = (p, v_p)$

Note s_f is smooth.

Thus we can think of a vector field as a function
$S : C^\infty(M) \to C^\infty(M)$, $S(f) = s_f$

Lemma 3.4.1: For any vector field s and smooth functions f and g on M, we have

$$s_{fg}(p) = f(p) \cdot s_g(p) + s_f(p) \cdot g(p)$$

Proof: $v_p(fg) = f(p)v_p(g) + v_p(f)g(p)$

Lemma 3.4.2: Let $S : C^\infty(M) \to C^\infty(M)$ be linear, and suppose $S(fg)(p) = f(p) \cdot S(g)(p) + S(f)(p) \cdot g(p)$. Then S is a vector field.

Proof: Define $S(p) : C^\infty(M) \to \mathbf{R}$ to be the function which sends $f \in C^\infty(M)$ to $S(f)(p)$, i.e, the function $S(f)$ evaluated at p.

Note that the hypothesis implies that $S(p)$ is linear and satisfies the Liebniz rule and hence is a derivation.

Defn: If A, B are vector fields, let $AB = A \circ B$

Defn: The *Lie Bracket* of vector fields A and B is $[A, B] = AB - BA : C^\infty(M) \to C^\infty(M)$.

Thm: The Lie bracket of vector fields is a vector field.