Randell's Submanifolds (2.3) = Boothby's Regular submanifold (III.5):

If $K$ is a submanifold of $M$, then $K$ has the subspace topology.
If $(\phi_\alpha, U_\alpha)$ is a chart for $M$ such that $\phi_\alpha(U_\alpha) = \Pi_1^m(-\epsilon, \epsilon)$ and $\phi_\alpha(U_\alpha \cap K) = \Pi_1^k(-\epsilon, \epsilon) \times \Pi_{k+1}^m\{0\}$ , then $(\phi_\alpha|_{U_\alpha \cap K}, U_\alpha \cap K)$, $\phi_\alpha|_{U_\alpha \cap K} : U_\alpha \cap K \to \Pi_1^k(-\epsilon, \epsilon)$ is a chart for $K$.

Prop: If $U^{open} \subset M^m$, then $U$ is an $m$-dimensional submanifold of $M$.

Prop: If $K$ is a submanifold of $M$, then $i : K \to M$, $i(k) = k$, the inclusion map is smooth.

Ex: Find a counterexample to the above if we replace the hypothesis $K$ is a submanifold of $M$ with $K \subset M$.

Prop: If $f : N \to M$ is smooth and if $H$ is a submanifold of $N$, then $f : H \to M$ is smooth

Ex: Find a counterexample to the above if we replace the hypothesis $H$ is a submanifold of $N$ with $H \subset N$.

Prop: If $f : N \to M$ is smooth and if $K$ is a submanifold of $M$ and if $f(N) \subset K$, then $f : N \to K$ is smooth.

Ex: Find a counterexample to the above if we replace the hypothesis $K$ is a submanifold of $M$ with $K \subset M$.

Boothy III.6 = Randell Chapter 1.3

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Defn: $G$ is a *topological group* if

1.) $(G, *)$ is a group

2.) $G$ is a topological space.

3.) $* : G \times G \to G$, $*(g_1, g_2) = g_1 * g_2$, and $In : G \to G$, $In(g) = g^{-1}$ are both continuous functions.

Defn: $G$ is a *Lie group* if

1.) $G$ is a group

2.) $G$ is a smooth manifold.

3.) $*$ and $In$ are smooth functions.

Ex: $Gl(n, \mathbf{R})$ = set of all invertible $n \times n$ matrices is a Lie group:

1.) $(Gl(n, \mathbf{R}), matrix\ multiplication)$ is a group

2.) $(Gl(n, \mathbf{R})$ is a smooth manifold.

3.) $*(Gl(n, \mathbf{R}) \times (Gl(n, \mathbf{R}) \to (Gl(n, \mathbf{R})$,

$$*(A, B) = AB \text{ and}$$

$In : (Gl(n, \mathbf{R}) \to (Gl(n, \mathbf{R})$

$In(A) = A^{-1}$ are smooth functions.

Ex: $(\mathbf{C} - \{\mathbf{0}\}, \cdot)$, is a Lie group.

Thm: If $G$ is a Lie group and $H$ is a submanifold, then $H$ is a Lie group.

Ex: $(S^1, \cdot)$

Ex: $G_1, G_2$ lie groups implies $G_1 \times G_2$ is a lie group.

Ex: $T^n = S^1 \times ... \times S^1$ is a Lie group.

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The following maps are diffeomorphisms:

$In : G \to G$, $In(g) = g^{-1}$.

For $a \in G$,

$L_a : G \to G$, $L_a(g) = ag$

$R_a : G \to G$, $R_a(g) = ga$

Ex: $O(n) = \{M \in GL(n, \mathbf{R}) \mid M^t M = I\}$ is a Lie group.

Ex: $Sl(n, \mathbf{R}) = \{M \in GL(n, \mathbf{R}) \mid det(M) = 1\}$ is a Lie group.

Defn: $F$ is a *homomorphism* of Lie groups if $F$ is an algebraic homomorphism of Lie groups and $F$ is smooth.

Ex: $F : GL(n, \mathbf{R}) \to \mathbf{R} - \{\mathbf{0}\}$, $F(M) = det(M)$ is a homomorphism.