Charts for S^2:

Stereographic projection:

$U_N = S^n - \{N = (1, 0, ..., 0)\}$.
$\phi_N : U_N \to R^n$,
$\phi_N(\mathbf{x}) = \frac{\mathbf{x}}{(1-x_0)}$

$U_S = S^n - \{S = (-1, 0, ..., 0)\}$.
$\phi_S : U_S \to R^n$,
$\phi_S(\mathbf{x}) = \frac{\mathbf{x}}{(1+x_0)}$

Pre-atlas for S^n: $\mathcal{A}_1 = \{(U_N, \phi_N), (U_S, \phi_S)\}$,

---

Projection:

$U_{zp} = \{(x, y, z) \in S^2 \mid z > 0\}$,
$\varphi_{zp} : U_{zp} \to \{(x, y) \mid x^2 + y^2 < 1\}$,
$\varphi_{zp}(x, y, z) = (x, y)$

$U_{yp} = \{(x, y, z) \in S^2 \mid y > 0\}$,
$\varphi_{yp} : U_{yp} \to \{(x, z) \mid x^2 + z^2 < 1\}$,
$\varphi_{yp}(x, y, z) = (x, z)$

$U_{xp} = \{(x, y, z) \in S^2 \mid x > 0\}$,
$\varphi_{xp} : U_{xp} \to \{(y, z) \mid y^2 + z^2 < 1\}$,
$\varphi_{xp}(x, y, z) = (y, z)$

$U_{zn} = \{(x, y, z) \in S^2 \mid z < 0\}$,
$\varphi_{zn} : U_{zn} \to \{(x, y) \mid x^2 + y^2 < 1\}$,
$\varphi_{zn}(x, y, z) = (x, y)$

$U_{yn} = \{(x, y, z) \in S^2 \mid y < 0\}$,
$\varphi_{yn} : U_{yn} \to \{(x, z) \mid x^2 + z^2 < 1\}$,
$\varphi_{yn}(x, y, z) = (x, z)$

$U_{xn} = \{(x, y, z) \in S^2 \mid x < 0\}$,
$\varphi_{xn} : U_{xn} \to \{(y, z) \mid y^2 + z^2 < 1\}$,
$\varphi_{xn}(x, y, z) = (y, z)$

Compatibility:

$\varphi_{zp} \circ \varphi_{yn}^{-1} : \varphi_{yn}(U_{zp} \cap U_{yn}) \to \varphi_{zp} U_{zp} \cap U_{yn})$.

$U_{zp} \cap U_{yn} = \{(x, y, z) \in S^2 \mid z > 0 \text{ and } y < 0\}$.

$\varphi_{yn}(U_{zp} \cap U_{yn}) = \{(x, z) \mid z > 0 \text{ and } x^2 + z^2 < 1\}$

$\varphi_{zp} \circ \varphi_{yn}^{-1}(x, z) = \varphi_{zp}(x, -(1 - x^2 - z^2)^{1/2}, z) = (x, -(1 - x^2 - z^2)^{1/2})$.

Pre-atlas for S^2: $\mathcal{A}_2 = \{(U_{zp}, \varphi_{zp}), ..., (U_{xn}, \varphi_{xn})\}$,

---

Polar coordinates

$(x, y) = (rcos\theta, rsin\theta),\ r \geq 0,\ \theta \in [0, 2\pi)$,

$(r, \theta) = (\sqrt{x^2 + y^2}, tan^{-1}(y/x)$

$U_\gamma = S^1 - \{(1, \gamma)\}$,
$\phi_{(1,\gamma)} : U_p \to (0, 2\pi)$
$\phi_{(1,\gamma)}(r, \theta) = \theta - \gamma \text{ mod } 2\pi$

Pre-atlas for S^1: $\{(U_\gamma, \phi_{(1,\gamma)} \mid \gamma \in S^1\}$

Spherical coordinates

$(x, y, z) = (rsin\phi cos\theta, rsin\phi sin\theta, rcos\phi),\ r \geq 0,\ \theta \in [0, 2\pi),\ \phi \in [0, \pi]$

$(r, \theta, \phi) = (\sqrt{x^2 + y^2 + z^2}, tan^{-1}(y/x), tan^{-1}(\sqrt{x^2 + y^2}/z))$.

$U_p = S^2 - \{p\}$,
$\phi_p : U_p \to \phi_p(U)$
$\phi_p(r, \theta, \phi) = (\theta, \phi)$

Cylindrical: $(x, y, z) = (rcos\theta, rsin\theta, z),\ r \geq 0,\ \theta \in [0, 2\pi),\ z \in R$

$(r, theta, z) = (\sqrt{x^2 + y^2}, tan^{-1}(y/x), z)$.

:

$G(p, N) = \{[g] \mid g^{smooth} : U \to N$, for some U^{open} such that $p \in U \subset M\}$

$G(p) = G(p, \mathbf{R})$

Let $\alpha : I \to M$ where $I =$ an interval $\subset \mathbf{R}$, $\alpha(0) = p$. Note $[\alpha] \in G[0, M]$

Directional derivative of $[g]$ in direction $[\alpha] =$

$$D_\alpha g = \frac{d(g \circ \alpha)}{dt}|_{t=0} \in \mathbf{R}$$

$T_p(M) = \{v : G(p) \to \mathbf{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

$v \in T_p(M)$ is called a *derivation*

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$, the *standard basis* for $T_p(M) = \{v_1, ..., v_m\}$, where $v_i = D_{\alpha_i}$ and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

If $v \in T_p(M)$, then $v = \Sigma_{i=1}^m a_i v_i$ where $a_i = v([\pi_i \circ \phi])$

Suppose $f^{smooth} : M \to N$, $f(p) = q$.

The *tangent (or differential) map,*

$$df_p : T_pM \to T_qN$$

$df_p(v) =$ the derivation which takes $[g] \in G(q)$ to the real number $v([g \circ f])$

I.e., df_p takes the derivation $(v : G(p) \to \mathbf{R}) \in T_pM$

to the derivation in T_qN which takes

the germ $[g] \in G(q)$ to the real number $v([g \circ f])$.