Ex: $P^n(\mathbf{R}) = \mathbf{R}P^n = \mathbf{R}P^n = (\mathbf{R}^{n+1} - \{\mathbf{0}\})/(\mathbf{x} \sim t\mathbf{x})$
$= n$-dimensional real projective space is a smooth manifold.

Claim: $\mathbf{R}P^n$ is 2nd countable.

We will use
Lemma: If $\sim$ is open and if X has a countable basis, then $X/\sim$ has a countable basis.

[We will define a map which takes $\mathbf{y} \in [\mathbf{x}]$ to $t\mathbf{y} \in [\mathbf{x}]$]

Let $\phi_t : \mathbf{R}^{n+1} - \{\mathbf{0}\} \to \mathbf{R}^{n+1} - \{\mathbf{0}\}$, $\phi_t(\mathbf{x}) = t\mathbf{x}$.

ϕ_t is invertible with inverse $\phi_t^{-1} = \phi_{\frac{1}{t}}$.

Since ϕ_t and ϕ_t^{-1} are C^1 (as well as C^∞), ϕ_t is a homeomorphism.

Let U be open in $\mathbf{R}^{n+1} - \{\mathbf{0}\}$. Then $\phi_t(U)$ is open in $\mathbf{R}^{n+1} - \{\mathbf{0}\}$.

Thus $\pi^{-1}([U]) = \cup_{t \in \mathbf{R}} \phi_t(U)$ is open in $\mathbf{R}^{n+1} - \{\mathbf{0}\}$.

Thus $[U]$ is open in $\mathbf{R}P^n$. Hence $\sim$ is open.

Since $\mathbf{R}^n$ is 2nd countable, $\mathbf{R}P^n$ is 2nd countable.

Claim $\mathbf{R}P^n$ is Hausdorff

We will use
Lemma: Let $\sim$ be open. Then $\{(x, y) \mid x \sim y\}$ is closed in $X \times X$ iff $X/\sim$ is Hausdorff.

[We will show that $\{(x, y) \mid x \sim y\} = f^{-1}(\{0\})$ for some continuous function f. $x \sim y$ implies $x_i = ty_i$. Thus $\frac{x_i}{y_i} = \frac{x_j}{y_j}$. Hence $x_i y_j - y_i x_j = 0$ for all i, j.]

Let $f : \mathbf{R}^{n+1} - \{\mathbf{0}\} \times \mathbf{R}^{n+1} - \{\mathbf{0}\} \to \mathbf{R}$,
$f(x_1, ..., x_n, y_1, ..., y_n) = \Sigma_{i \neq j}(x_i y_j - y_i x_j)^2$.

f is C^1 (all partials of f exist and are continuous). Thus f is continuous.

Suppose $\mathbf{y} = t\mathbf{x}$, then
$f(x_1, ..., x_n, y_1, ..., y_n) = \Sigma_{i \neq j}(x_i t x_j - t x_i x_j)^2 = 0$.

Suppose $f(x_1, ..., x_n, y_1, ..., y_n) = \Sigma_{i \neq j}(x_i y_j - y_i x_j)^2 = 0$. Then $x_i y_j - y_i x_j = 0$ for all i, j. Since $\mathbf{x} \neq \mathbf{0}$, there exists i_0 such that $x_{i_0} \neq 0$. Thus $y_j = \frac{y_{i_0}}{x_{i_0}} x_j$ and $\mathbf{y} = \frac{y_{i_0}}{x_{i_0}} \mathbf{x}$. Hence $\mathbf{x} \sim \mathbf{y}$.

Hence $f^{-1}(\{0\}) = \{(x, y) \mid x \sim y\}$. Since f is continuous and $\{0\}$ is closed in $\mathbf{R}$, $\{(x, y) \mid x \sim y\}$ is closed in $\mathbf{R}^{n+1} - \{\mathbf{0}\} \times \mathbf{R}^{n+1} - \{\mathbf{0}\}$. Thus $\mathbf{R}^{n+1} - \{\mathbf{0}\}/\sim$ is Hausdorff.

We will show that $\mathbf{R}P^n$ is locally Euclidean by finding a (pre) atlas:

Let $V_i = \{x \in \mathbf{R}^{n+1} - \{\mathbf{0}\} \mid x_i \neq 0\} \subset \mathbf{R}^{n+1} - \{\mathbf{0}\}$

Let $F_i : V_i \to \mathbf{R}^n$, $F_i(x_1, ..., x_{n+1}) = (\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i})$
$= \frac{1}{x_i}(x_1, ..., \hat{x}_i, ..., x_{n+1})$

$$F_i(t\mathbf{x}) = (\frac{tx_1}{tx_i}, \frac{tx_2}{tx_i}, ..., \frac{tx_{i-1}}{tx_i}, \frac{tx_{i+1}}{tx_i}, ..., \frac{tx_{n+1}}{tx_i}) = (\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i}) = F_i(\mathbf{x}).$$

Let $U_i = \pi(V_i)$ Then $\phi_i : U_i \to \mathbf{R}^n$, $\phi_i([\mathbf{x}]) = F_i(\mathbf{x})$ is well-defined.

Claim: (ϕ_i, U_i) is a chart.

Subclaim 1: U_i is open in $\mathbf{R}P^n$.

$\pi_i^{-1}(U_i) = \pi_i^{-1}(\pi(V_i)) = V_i$ [by set theory]. Hence U_i is open in $\mathbf{R}P^n$.

Subclaim 2: $\phi(U_i)$ is open in $\mathbf{R}^n$.

Claim: ϕ_i is onto.

Let $(x_1, ..., x_n) \in \mathbf{R}^n$. $\phi_i([(x_1, ..., x_{i-1}, 1, x_i, .., x_n]) = (x_1, ..., x_n)$. Thus ϕ_i is onto.

Since $\phi(U_i) = \mathbf{R}^n$, $\phi(U_i)$ is open in $\mathbf{R}^n$.

Subclaim 3: ϕ_i is a homeomorphism.

Claim: ϕ_i is continuous.

Let V be open in $\mathbf{R}^n$.
Since $F_i \in C^1$, $F_i^{-1}(V)$ is open in $\mathbf{R}^{n+1} - \{\mathbf{0}\}$.

$\pi^{-1} \circ \phi_i^{-1}(V) = F_i^{-1}(V)$. Thus $\phi_i^{-1}(V)$ is open in U_i and ϕ_i is continuous.

Claim: ϕ_i is 1:1.

If $\phi_i([\mathbf{x}]) = \phi_i([\mathbf{y}])$, then $\frac{x_j}{x_i} = \frac{y_j}{y_i}$ for all j. Thus $y_i = \frac{y_i}{x_i} x_j$ and thus $\mathbf{y} = \frac{y_i}{x_i}\mathbf{x}$. Thus ϕ_i is 1:1.

Since ϕ_i is 1:1 and onto, ϕ_i^{-1} exists.

Claim: $\phi_i^{-1} : \mathbf{R}^n \to \mathbf{R}P^n$ is continuous.

$\phi_i^{-1}(x_1, ..., x_n) = [(x_1, ..., x_{i-1}, 1, x_i, .., x_n)]$.

Let $f_i : \mathbf{R}^n \to \mathbf{R}^{n+1}$, $f_i(\mathbf{x}) = (x_1, ..., x_{i-1}, 1, x_i, ..., x_n)$. f_i is C^1 and hence continuous.

$\pi \circ f(\mathbf{x}) = [(x_1, ..., x_{i-1}, 1, x_i, .., x_n)] = \phi_i^{-1}(x_1, ..., x_n)$.

Since π and f are continuous, $\phi_i^{-1} : \mathbf{R}^n \to \mathbf{R}P^n$ is continuous.

Thus (ϕ_i, U_i) is a chart.

Claim: $\{(\phi_i, U_i) \mid i = 1, ..., n+1\}$ is a (pre) atlas for $\mathbf{R}P^n$.

$\mathbf{R}^{n+1} - \{\mathbf{0}\} = \cup_{i=1}^{n+1} V_i$. Thus $\mathbf{R}P^n = \cup_{i=1}^{n+1} U_i$.

Claim: $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is smooth.

Suppose $j < i$.

$$\begin{aligned}\phi_j(\phi_i^{-1}(x_1, ..., x_n)) &= \phi_j([x_1, ..., x_{i-1}, 1, x_i, ..., x_n]) \\ &= \left(\tfrac{x_1}{x_j}, ..., \tfrac{x_{i-1}}{x_j}, \tfrac{1}{x_j}, \tfrac{x_i}{x_j}, ..., \tfrac{x_n}{x_j}\right)\end{aligned}$$

Since all the components of $\phi_j \circ \phi_i^{-1}$ are rational functions with non-vanishing denominators ($\mathbf{0}$ is not in the domain of $\phi_j \circ \phi_i^{-1}$), $\phi_j \circ \phi_i^{-1}$ is smooth.

Similarly $\phi_j \circ \phi_i^{-1}$ is smooth when $j > i$.

Thus $\{(\phi_i, U_i) \mid i = 1, ..., n+1\}$ is a (pre) atlas for $\mathbf{R}P^n$, and $\mathbf{R}P^n$ is a smooth manifold.