2.7

Let $A_{m\times n} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ ... \\ \mathbf{r}_m \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 & ... & \mathbf{c}_n \end{pmatrix}$

Rank of $A = dim(span\{\mathbf{r}_1, ..., \mathbf{r}_m\}) = dim(span\{\mathbf{c}_1, ..., \mathbf{c}_m\})$
$=$ maximum order of any nonvanishing minor determinant.

Ex: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$

Let $F : U \subset \mathbf{R}^n \to \mathbf{R}^m \in C^1$.

Rank of F at $x =$ rank of $DF(x)$

F has rank k if F has rank k at each x.

$Det : M^{n\times m} \to \mathbf{R}$ is a continuous function.

Suppose rank $DF(a) = k$ implies there exists V open such that $a \in V$ and $DF(x) \geq k$ for all $x \in V$

Ex: $F(x_1, x_2) = (x_1x_2 + 5, x_1 + x_2 - 3)$

$DF = \begin{pmatrix} x_2 & x_1 \\ 1 & 1 \end{pmatrix}$

Rank Theorem: Suppose $A_0 \subset \mathbf{R}^n$, $B_0 \subset \mathbf{R}^m$, $F : A_0 \to B_0 \in C^1$ $a \in A_0, b \in B_0$. Suppose rank F $= k$.

Then there exists $A^{open} \subset A_0$ such that $a \in A$ and $B^{open} \subset B_0$ such that $b \in B$ and G, H, C^r diffeomorphisms

such that $G : A \to U^{open} \subset \mathbf{R}^n$, $H : B \to V^{open} \subset \mathbf{R}^m$ and

$$H \circ F \circ G^{-1}(x_1, ..., x_n) = (x_1, ..., x_k, 0, ..., 0)$$