HW 3 part 2

Let $H_1(\mathbf{x}) = F(\mathbf{x}+\mathbf{a}) - F(\mathbf{a})$. Then $DH_1(\mathbf{0}) =$

Let $H(\mathbf{x}) = [DF(\mathbf{a})]^{-1}H_1(\mathbf{x})$

$H(\mathbf{0}) =$ and by the chain rule, $DH(\mathbf{0}) =$

By 8, 9, and note, H is a diffeomorphism iff F is a diffeomorphism.

Thm 6.4 (Inverse Function Theorem): Suppose $F : W^{open} \subset \mathbf{R}^n \to R^n \in C^r$. Suppose for $\mathbf{a} \in W$, $det(DF(a)) \neq 0$. Then there exists U such that $\mathbf{a} \in U^{open}$, $V = F(U)$ is open, and $F : U \to V$ is a C^r-diffeomorphism. Moreover, for $\mathbf{x} \in U$ and $\mathbf{y} = F(\mathbf{x})$, $DF^{-1}_{\mathbf{y}} = (DF_{\mathbf{x}})^{-1}$

Proof:

Without loss of generality, assume $F(\mathbf{0}) = \mathbf{0}$ and $DF(\mathbf{0}) = I$

Claim: There exists a nbhd U_1 of $\mathbf{0}$ such that $F|_{U_1}$ is 1:1.

Method 1: Suppose for all $n \in \mathbf{N}$, there exists an $0 < \mathbf{h}_n < \frac{1}{n}$ such that $F(\mathbf{h}_n) = F(\mathbf{0})$

Then $\frac{F(\mathbf{0}+\mathbf{h}_n)-F(\mathbf{0})-I\mathbf{h}_n}{||\mathbf{h}_n||} =$

But $lim_{\mathbf{h}\to\mathbf{0}} \frac{F(\mathbf{0}+\mathbf{h})-F(\mathbf{0})-I\mathbf{h}}{||\mathbf{h}||} =$

Hence

Claim: $DF^{-1}_{\mathbf{y}} = (DF_{\mathbf{x}})^{-1}$ for $\mathbf{y}$ in some nbhd V_1 of $\mathbf{0}$.

Let $\mathbf{c} \in F^{-1}(V_1)$ and let $T = DF(c)$. Since F is differentiable, there exists $r(\mathbf{x}, \mathbf{c})$ such that

Let $\mathbf{y} = F(\mathbf{x})$ and $\mathbf{d} = F(\mathbf{c})$.

Then $\mathbf{y} - \mathbf{d} = T[F^{-1}(\mathbf{y}) - F^{-1}(\mathbf{d})] + ||F^{-1}(\mathbf{y}) - F^{-1}(\mathbf{d})||r(F^{-1}(\mathbf{y}), F^{-1}(\mathbf{d}))$

Thus $F^{-1}(\mathbf{y}) - F^{-1}(\mathbf{d}) = T^{-1}(\mathbf{y} - \mathbf{d}) - ||F^{-1}(\mathbf{y}) - F^{-1}(\mathbf{d})||T^{-1}r(F^{-1}(\mathbf{y}), F^{-1}(\mathbf{d}))$

Let $R(\mathbf{y}, \mathbf{d}) = \frac{F^{-1}(\mathbf{y})-F^{-1}(\mathbf{d})-T^{-1}(\mathbf{y}-\mathbf{d})}{||\mathbf{y}-\mathbf{d}||} = -\frac{||F^{-1}(\mathbf{y})-F^{-1}(\mathbf{d})||}{||\mathbf{y}-\mathbf{d}||}T^{-1}r(F^{-1}(\mathbf{y}), F^{-1}(\mathbf{d}))$

IF F^{-1} is continuous, then $lim_{\mathbf{y}\to\mathbf{d}} r(F^{-1}(\mathbf{y}), F^{-1}(\mathbf{d})) =$

IF $\frac{||F^{-1}(\mathbf{y})-F^{-1}(\mathbf{d})||}{||\mathbf{y}-\mathbf{d}||}$ is bounded, then $lim_{\mathbf{y}\to\mathbf{d}} R(\mathbf{y}, \mathbf{d}) =$

Hence $DF^{-1}(\mathbf{d}) =$

Thus we need to show F^{-1} is continuous and $\frac{||F^{-1}(\mathbf{y})-F^{-1}(\mathbf{d})||}{||\mathbf{y}-\mathbf{d}||}$ is bounded.