HW 3:

Let $F : \mathbf{R} \to \mathbf{R}^2$, $F(x) = (2,3)x$
$G : \mathbf{R}^2 \to \mathbf{R}^3$, $G(x,y) = (xy, x^2, x + 2y + 5)$
$H : \mathbf{R} \to \mathbf{R}^2$, $H(x) = (x^2, x^3)$
$k : \mathbf{R}^2 \to \mathbf{R}$, $k(x,y) = x^8 + 5xy$.

1.) Use the chain rule to calculate $D(G \circ F)_2$

2.) Use the product rule to calculate $D(FH)_2$

3.) Let $\mathbf{a} = (3,4)$. Let $X_a = 9E_{1\mathbf{a}} - E_{2\mathbf{a}}$. Then $X_a^*(k) =$

F is a C^r-*diffeomorphism* if

(1) F is a homeomorphism

(2) $F, F^{-1} \in C^r$

F is a *diffeomorphism* if F is a C^∞-diffeomorphism.

4.) Give an example of a homeomorphism which is analytic (ie C^∞ and near a, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$, its Taylor series) which is not a diffeomorphism.

5.) Suppose $F'(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in U^{open} \subset \mathbf{R}^n$. Show F cannot be a homeomorphism. What can you say about F (hint: MVT).

6.) Suppose $f : R \to R$, $f'(x) \neq 0$ for all $x \in U$. Show that the derivative of f^{-1} exists for all $y \in f(U)$

7.) Suppose F is a C^1-diffeomorphism. Show that DF_x is nonsingular (ie $det(DF_x) \neq 0) \forall x \in dom(F)$

8.) Ex 1: Show $F : \mathbf{R}^n \to \mathbf{R}^n$, $F(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ is a diffeomorphism.

9.) Ex 2: Determine when $F : \mathbf{R}^n \to \mathbf{R}^m$, $F(\mathbf{x}) = Ax$, where A is an $m \times n$ matrix, is a diffeomorphism. $DF_x =$.

Note that if F and G are diffeomorphism, then $F \circ G$ is a diffeomorphism (when $F \circ G$ is defined).

Thm 6.5 (Contracting mapping theorem): Let M be a complete metric space and let $T : M \to M$. Suppose there exists a constant $\lambda \in [0,1)$ such that for all $x, y \in M$, $d(T(x), T(y)) \leq \lambda d(x,y)$. Then T has a unique fixed point.

Proof: See class notes (Recall $T^n(x_0)$ is a Cauchy sequence. Since M be a complete metric space, $T^n(x_0)$ converges, say to a. Then $d(T(a), a) = 0$).

Thm 6.4 (Inverse Function Theorem): Suppose $F : W^{open} \subset \mathbf{R}^n \to R^n \in C^r$. Suppose for $a \in W$, $det(DF_a) \neq 0$. Then there exists U such that $a \in U^{open}$, $V = F(U)$ is open, and $F : U \to V$ is a C^r-diffeomorphism. Moreover, for $x \in U$ and $y = F(x)$, $DF_y^{-1} = (DF_x)^{-1}$