$T_{\mathbf{a}}(\mathbf{R}^n) = \{(\mathbf{a}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{R}^n\}$

$\phi(\mathbf{ax}) = \mathbf{x} - \mathbf{a}$

canonical basis $\{E_{i\mathbf{a}} = \phi^{-1}(e_i) \mid i = 1, ..., n\}$

Let $C^\infty(a) = \{f : X \subset \mathbf{R}^n \to \mathbf{R} \in C^\infty \mid a \in domf\}$

$f \sim g$ if $\exists U^{open}$ s.t. $\mathbf{a} \in U$ and $f(x) = g(x) \forall x \in U$.

$f_i : U_i \to \mathbf{R} \in C^\infty(a)$ implies $f_1 + f_2 : U_1 \cap U_2 \to \mathbf{R} \in C^\infty(a)$ and $\alpha f_i : U_i \to \mathbf{R} \in C^\infty(a)$

Thus $C^\infty(a)$ is an algebra over $\mathbf{R}$

Let $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$

$X_{\mathbf{a}}^* : C^\infty(\mathbf{a}) \to \mathbf{R}$

$X_{\mathbf{a}}^*(f) = \Sigma_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}_{\mathbf{a}}$

Let $x_j : \mathbf{R}^n \to \mathbf{R}$, $x_j(\mathbf{x}) = x_j$

$X_{\mathbf{a}}^*(x_j) = \Sigma_{i=1}^n \xi_i \frac{\partial x_j}{\partial x_i}_{\mathbf{a}} = \xi_i$

$X_{\mathbf{a}}^*$ is linear and satisfies the Leibniz rule.

Let $\mathcal{D}(a) = \{D : C^\infty(\mathbf{a}) \to \mathbf{R} \mid D$ is linear and satisfies the Leibniz rule $\}$

Define $(\alpha D_1 + \beta D_2)(f) = \alpha[D_1(f)] + \beta[D_2(f)]$

$\mathcal{D}(a)$ is closed under addition and scalar multiplication and hence is a vector space over $\mathbf{R}$

Let $j : T_{\mathbf{a}}(\mathbf{R}^n) \to \mathcal{D}(a)$, $j(X_{\mathbf{a}}) = X^*_{\mathbf{a}}$

Claim: j is an isomorphism.

Let $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$ and $Z_{\mathbf{a}} = \Sigma_{i=1}^n \zeta_i E_{i\mathbf{a}}$

j is a homomorphism.

j is 1-1:

If $j(X_{\mathbf{a}}) = j(Z_{\mathbf{a}})$, then $X^*_{\mathbf{a}}(x_j) = \Sigma_{i=1}^n \xi_i \frac{\partial x_j}{\partial x_i}_{\mathbf{a}} = \xi_i = \zeta_i = Z^*_{\mathbf{a}}(x_j)$

j is onto:

Let D be a derivation.

Suppose $f(\mathbf{x}) = 1$. Then $Df = 0$

Suppose $g(\mathbf{x}) = c$. Then $Dg = D(cf) = cDf = 0$

Let $h_i(\mathbf{x}) = x_i$. Let $\xi_i = Dh_i$. Then $D = X^*_{\mathbf{a}}$ where $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$ (proof: long calculation, see Boothby).

Note since $X^*_{\mathbf{a}}(f) = \Sigma_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}_{\mathbf{a}}$, $j(E_{i\mathbf{a}}) =$